Cycles in the chamber homology of GL(3)

Anne-Marie Aubert, Samir Hasan and Roger Plymen

Abstract

Let F be a nonarchimedean local field and let $\mathrm{GL}(N) = \mathrm{GL}(N, F)$. We prove the existence of parahoric types for $\mathrm{GL}(N)$. We construct representative cycles in all the homology classes of the chamber homology of GL(3).

1 Introduction

Let F be a nonarchimedean local field and let $G = \mathrm{GL}(N) = \mathrm{GL}(N, F)$. The enlarged building $\beta^1 G$ of G is a polysimplicial complex on which G acts properly. We select a chamber $C \subset \beta^1 G$. This chamber is a polysimplex, the product of an n-simplex by a 1-simplex:

$$C = \Delta_n \times \Delta_1.$$

To this datum we will attach a *homological coefficient system*, see [13, p.11]. To each simplex $x \in \Delta_n$ we attach the representation ring $R(G(x))$ of the stabilizer $G(x)$, and to each inclusion $x \to y$ we attach the induction map:

$$\mathrm{Ind}_{G(x)}^{G(y)} : R(G(x)) \to R(G(y)).$$

This creates the homology of the simplicial set Δ_n with the above coefficient system. The resulting homology groups are denoted $h_j(G), 0 \leq j \leq N-1$.

For each point $\mathfrak{s}$ in the Bernstein spectrum $\mathfrak{B}(G)$ (see appendix B) we will select an $\mathfrak{s}$-type (J, τ). Here, J denotes a certain compact open subgroup of G, and τ denotes a certain irreducible smooth representation of J, see [8, 9, 10].

The following result is due to Bushnell-Kutzko [8, 9, 10].

Theorem 1. Existence of types. *Let $\mathfrak{s} \in \mathfrak{B}(G)$. There exists an $\mathfrak{s}$-type (J, τ).*

Let $\mathfrak{s} \in \mathfrak{B}(G)$. An $\mathfrak{s}$-type (J, λ) will be called *parahoric* if J is a parahoric subgroup of G.

Our first result is the following theorem.

Theorem 2. Existence of parahoric types. *Let $\mathfrak{s} \in \mathfrak{B}(G)$. Then there exists a parahoric $\mathfrak{s}$-type $(J^{\mathfrak{s}}, \tau)$.*

The parahoric subgroup $J^{\mathfrak{s}}$ only depends on certain invariants attached to $\mathfrak{s}$. For details of these invariants, see appendix D.

In the proof of Theorem 2, we have to call upon several of the technical resources developed by Bushnell-Kutzko.

We now specialize to GL(3). In this article, we will explicitly construct representative cycles in *all* the homology classes in $h_0(G) \oplus h_1(G) \oplus h_2(G)$ when $G = \mathrm{GL}(3)$. This allows us to compute the chamber homology groups of GL(3) according to the following formulas:

$$\mathrm{H}_{\mathrm{ev}}(G; \beta^1 G) = h_0(G) \oplus h_1(G) \oplus h_2(G) = \mathrm{H}_{\mathrm{odd}}(G; \beta^1 G).$$

We will demonstrate that each parahoric $\mathfrak{s}$-type $(J^{\mathfrak{s}}, \tau)$ creates finitely many cycles in $h_0(G) \oplus h_1(G) \oplus h_2(G)$. To prove that all homology classes in $h_0(G) \oplus h_1(G) \oplus h_2(G)$ are thereby accounted for, we invoke the K-theory of the reduced C^*-algebra $\mathcal{A} := C_r^*(G)$. The K-theory is torsion-free [17].

The abelian groups $\mathrm{H}_{\mathrm{ev/odd}}(G; \beta^1 G)$ and $K_j(\mathcal{A})$ admit compatible Bernstein decompositions, see appendix B. This leads, for each $\mathfrak{s} \in \mathfrak{B}(G)$, to the equalities

$$\operatorname{rank} \mathrm{H}_{\mathrm{ev/odd}}(G; \beta^1 G)^{\mathfrak{s}} = \operatorname{rank} K_0(\mathcal{A}^{\mathfrak{s}}) = \operatorname{rank} K_1(\mathcal{A}^{\mathfrak{s}}). \tag{1}$$

The ranks of the finitely generated abelian groups on the right-hand-side are easily computed (see appendix C).

Theorem 3. *Let $G = \mathrm{GL}(3)$, and let $\mathfrak{s} = [M, \sigma]_G$. Each parahoric $\mathfrak{s}$-type $(J^{\mathfrak{s}}, \tau)$ creates finitely many cycles in $h_0(G) \oplus h_1(G) \oplus h_2(G)$, and all homology classes in $h_0(G) \oplus h_1(G) \oplus h_2(G)$ are thereby accounted for. Quite specifically, we have*

- *if $M = \mathrm{GL}(3)$ then*

$$\mathrm{H}_{\mathrm{ev}}(G; \beta^1 G)^{\mathfrak{s}} = \mathbb{Z} = \mathrm{H}_{\mathrm{odd}}(G; \beta^1 G)^{\mathfrak{s}}$$

- *if $M = \mathrm{GL}(2) \times \mathrm{GL}(1)$ then*

$$\mathrm{H}_{\mathrm{ev}}(G; \beta^1 G)^{\mathfrak{s}} = \mathbb{Z}^2 = \mathrm{H}_{\mathrm{odd}}(G; \beta^1 G)^{\mathfrak{s}}$$

- *if* $M = \mathrm{GL}(1) \times \mathrm{GL}(1) \times \mathrm{GL}(1)$ *then*

$$\mathrm{H}_{\mathrm{ev}}(G; \beta^1 G)^{\mathfrak{s}} = \mathbb{Z}^4 = \mathrm{H}_{\mathrm{odd}}(G; \beta^1 G)^{\mathfrak{s}}$$

From this point of view, the types for GL(3) exceed their original expectations. Let $\widehat{\mathcal{A}^{\mathfrak{s}}}$ denote the dual of the C^*-algebra $\mathcal{A}^{\mathfrak{s}}$. This is a compact Hausdorff space. Since K-theory for unital C^*-algebras is compatible with topological K-theory of compact Hausdorff spaces, we have

$$K_j(\mathcal{A}^{\mathfrak{s}}) \cong K^j(\widehat{\mathcal{A}^{\mathfrak{s}}}).$$

Therefore, the $\mathfrak{s}$-type also computes the topological K-theory of the compact space $\widehat{\mathcal{A}^{\mathfrak{s}}}$. The space $\widehat{\mathcal{A}^{\mathfrak{s}}}$ is precisely the space of all those tempered representations of GL(3) which have inertial support $\mathfrak{s}$.

Sections $4-6$ are devoted to a proof of Theorem 2, and sections $7-9$ are devoted to a proof of Theorem 3.

Preliminary work in the direction of Theorem 3 was done with Paul Baum and Nigel Higson, and recorded in [4]. The diagrams in [4] are relevant to the present article. In [4] all computations were in the *tame* case. We confront here the general case: this is much more technical. We require much detailed information in the theory of types; in particular we need detailed information concerning *compact* intertwining sets.

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2 General results on types

We will collect here some general results on types which will used in the paper. In this section G denotes the group of F-points of an arbitrary reductive connected algebraic group $\mathbf{G}$ defined over F.

Let $\mathfrak{R}(G)$ denote the category of smooth complex representations of G. Recall that, for each irreducible smooth representation π of G, there exists a Levi subgroup L of a parabolic subgroup P of G and an irreducible supercuspidal representation σ of L such that π is equivalent to a subquotient of the parabolically induced representation $I_P^G(\sigma)$. The pair (L, σ) is unique up to conjugacy and the inertial class $\mathfrak{s} = [M, \sigma]_G$ (see appendix B) is called the *inertial support* of π.

We have the standard decomposition (see [5, (2.10)])

$$\mathfrak{R}(G) = \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G) \tag{2}$$

into full sub-categories, where the objects of $\mathfrak{R}^{\mathfrak{s}}(G)$ are those smooth representations of G all of whose irreducible subquotients have inertial support $\mathfrak{s}$.

Let $\mathfrak{s}$ be a point in the Bernstein spectrum of G, and let (J, τ) be an $\mathfrak{s}$-type, *i.e.,* τ is an irreducible smooth representation of an open compact subgroup J of G such that for any irreducible smooth representation π of G, the restriction of π to J contains τ if and only if π is an object of $\mathfrak{R}^{\mathfrak{s}}(G)$, [9, (4.2)]. When $G = \mathrm{GL}(N, F)$, it has been proved [8, 10] that there exists an $\mathfrak{s}$-type for each point $\mathfrak{s}$ in $\mathfrak{B}(G)$.

Proposition 1. *Let $K \supset J$ be an open compact subgroup of G such that $\alpha := \mathrm{Ind}_J^K \tau$ is irreducible. Then (K, α) is an $\mathfrak{s}$-type.*

Proof. Let π be an irreducible smooth representation of G. Using Frobenius reciprocity, we see that

$$\mathrm{Hom}_K \left(\alpha, \mathrm{Res}_K^G(\pi)\right) = \mathrm{Hom}_J \left(\tau, \mathrm{Res}_J^G(\pi)\right).$$

The result follows from the definition of $\mathfrak{s}$-types. □

Let J, J', K be subgroups of G with J, J' compact open and $J \subset K$, $J' \subset K$. Let λ, λ' be representations of J, J' on finite-dimensional vector spaces V, V'. Let $g \in G$. Then $gJg^{-1} \cap J'$ is a subgroup of J'. We set ${}^g\lambda(x) := \lambda(g^{-1}xg)$. We define the *$g$-intertwining vector space* of (λ, λ') to be

$$\mathcal{I}_g(\lambda, \lambda') = \mathrm{Hom}_{gJg^{-1} \cap J'}({}^g\lambda, \lambda').$$

We will write $\mathcal{I}_g(\lambda) = \mathcal{I}_g(\lambda, \lambda)$.

Definition 1. (1) We say that g *intertwines* λ if $\mathcal{I}_g(\lambda) \neq 0$. The *$K$-intertwining set of* λ is

$$\mathcal{I}_K(\lambda) = \{g \in K \,:\, \mathcal{I}_g(\lambda) \neq 0\} \subset K.$$

(2) We say that g *intertwines* λ *and* λ' if $\mathcal{I}_g(\lambda, \lambda') \neq 0$. The *$K$-intertwining set of* λ *and* λ' is

$$\mathcal{I}_K(\lambda, \lambda') = \{g \in K \,:\, \mathcal{I}_g(\lambda, \lambda') \neq 0\} \subset K.$$

In [9, 10, 8], the results centre around identification of the G-intertwining set $I_G(\lambda)$. In our applications, we shall need only the K-intertwining set $I_K(\lambda)$ where K is compact.

In order to study the induced representations and their decomposition into irreducible constituents, we need to use the Mackey formulas repeatedly.

We assume now that K is open compact. Then J, J' have finite index in K. We have the Mackey formula:

$$\mathrm{Hom}_K\left(\mathrm{Ind}_J^K(\lambda),\ \mathrm{Ind}_{J'}^K(\lambda')\right) \cong \bigoplus \mathcal{I}_x(\lambda, \lambda') \tag{3}$$

with $x \in J\backslash K/J'$. If $\lambda = \lambda \cong \lambda'$ then we set $\mathcal{I}_g(\lambda) = \mathcal{I}_g(\lambda, \lambda')$ and we then have the isomorphism of $\mathbb{C}$-vector spaces

$$\mathrm{End}_K\left(\mathrm{Ind}_J^K(\lambda)\right) \cong \bigoplus \mathcal{I}_x(\lambda)$$

with $x \in J\backslash K/J$.

The following is an immediate consequence: we will use this result repeatedly.

Proposition 2. *If* $\mathcal{I}_K(\lambda) = J$ *then* $\mathrm{Ind}_J^K(\lambda)$ *is irreducible.*

We will use the following immediate result.

Proposition 3. *If* $\mathrm{Ind}_J^K(\lambda)$ *and* $\mathrm{Ind}_{J'}^K(\lambda')$ *are irreducible, then*

$$\mathrm{Ind}_J^K(\lambda) \cong \mathrm{Ind}_{J'}^K(\lambda') \Longleftrightarrow \mathcal{I}_K(\lambda, \lambda') = JyJ'$$

for some element y.

Proposition 4. *Let* $(J^{\mathfrak{s}}, \tau^{\mathfrak{s}})$ *be an* $\mathfrak{s}$*-type,* $(J^{\mathfrak{s}'}, \tau^{\mathfrak{s}'})$ *be a* $\mathfrak{s}'$*-type with* $\mathfrak{s}$*,* $\mathfrak{s}'$ *in* $\mathfrak{B}(G)$*,* $\mathfrak{s} \neq \mathfrak{s}'$*. Let* J *be a compact open subgroup of* G *such that* $J^{\mathfrak{s}} \subset J$*,* $J^{\mathfrak{s}'} \subset J$*. Then we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_J(\mathrm{Ind}_{J^{\mathfrak{s}}}^J \tau^{\mathfrak{s}}, \mathrm{Ind}_{J^{\mathfrak{s}'}}^J \tau^{\mathfrak{s}'}) = 0.$$

Proof. From the Mackey formula (3), it is equivalent to prove that $\mathcal{I}_J(\tau^{\mathfrak{s}}, \tau^{\mathfrak{s}'}) = 0$. The proof of the equivalence of (i) and (ii) of [9, Theorem 9.3.a] shows that $\mathfrak{s} = \mathfrak{s}'$ if and only if $\mathcal{I}_G(\tau^{\mathfrak{s}}, \tau^{\mathfrak{s}'}) \neq 0$. The result follows. $\square$

Let J be a compact open subgroup of G and $(\tau, \mathcal{W})$ be an irreducible smooth representation of J. Let $(\tau^\vee, \mathcal{W}^\vee)$ be the contragredient representation of $(\tau, \mathcal{W})$.

For any subgroup K of G, let $\mathcal{H}(K, \tau)$ denote the space of compactly supported functions $f\colon K \to \mathrm{End}_{\mathbb{C}}(\mathcal{W}^\vee)$ such that $f(j_1 k j_2) = \tau^\vee(j_1) f(k) \tau^\vee(j_2)$, for any $j_i \in J$, $k \in K$. The standard convolution operation gives $\mathcal{H}(K, \tau)$ the structure of an associative unital $\mathbb{C}$-algebra.

Let M be a Levi subgroup of G, and let (J_M, τ_M) be a $\mathfrak{t}$-type, with $\mathfrak{t} := [M, \sigma]_M$ a (supercuspidal) point of the Bernstein spectrum of M.

We recall from [9, Definition 8.1] that the pair (J, τ) is a *G-cover of* (J_M, τ_M) if $J \cap M = J_M$ and $\tau_{|J_M} \cong \tau_M$, and if the following conditions hold for every parabolic subgroup P of G with Levi subgroup M:

(1) (J,τ) it is *decomposed with respect to* (M,P), that is, J admits the Iwahori decomposition:

$$J = J \cap U \cdot J_M \cdot J \cap \overline{U},$$

and the groups $J \cap U$, $J \cap \overline{U}$ are both contained in the kernel of τ (here U, $\overline{U}$ denote the unipotent radicals of P and of its opposite parabolic subgroup, respectively),

(2) there exists an invertible element of $\mathcal{H}(G,\tau)$ supported on a double coset Jz_PJ, where z_P is a central element in M, which is strongly (P,J)-*positive* in the sense of [9, Definition (6.16)].

The group $\Psi(M)$ of unramified quasicharacters of M has the structure of a complex torus. The action (by conjugation) of $\mathrm{N}_G(M)$ on M induces an action of $W(M) := \mathrm{N}_G(M)/M$ on $\mathfrak{B}(M)$. Let $W_\mathfrak{t}$ denote the stabilizer of $\mathfrak{t} = [M,\sigma]_M$ in $W(M)$. Thus $W_\mathfrak{t} = N_\mathfrak{t}/M$, where

$$\mathrm{N}_\mathfrak{t} = \{n \in \mathrm{N}_G(M) \, : \, {}^n\sigma \cong \nu\sigma, \quad \text{for some } \nu \in \Psi(M)\} \tag{4}$$

denotes the $\mathrm{N}_G(M)$-normalizer of $\mathfrak{t}$.

We will need the following Proposition which gives a bound for the compact intertwining.

Proposition 5. [11] *We assume here that $G = \mathrm{GL}(N,F)$. Let M be a Levi subgroup of G, let (J,τ) be a G-cover of a $\mathfrak{t}$-type, with $\mathfrak{t} = [M,\sigma]_M$ a point of the Bernstein spectrum of M, and let K be a compact subgroup of G which contains J. Let t denote the number of double classes $J\backslash K/J$ which intertwine τ. Then*

$$t \le |W_\mathfrak{t}|.$$

Proof. It is a classical result that t is bounded by the dimension of $\mathcal{H}(K,\tau)$. The hypotheses on the the supercuspidal representation σ which are listed in [11, §1.3] are identical to those listed in [9, (5.5)]. Since $G = \mathrm{GL}(N,F)$, it follows from [9, Comments (b) and (d) on (5.5)] that these hypotheses are satisfied, and so we can apply [11, Theorem 1.5(ii)]. We infer that

$$\dim_\mathbb{C} \mathcal{H}(K,\tau) \le |W_\mathfrak{t}|.$$

□

3 Chamber homology groups

Let $\mathfrak{o}_F$ denote the ring of integers of F, let $\varpi = \varpi_F$ be a uniformizer in F, and $\mathfrak{p}_F = \varpi_F \mathfrak{o}_F$ denote the maximal ideal of $\mathfrak{o}_F$. We set

$$\Pi = \Pi_N = \begin{pmatrix} 0 & \mathrm{I}_{N-1} \\ \varpi & 0 \end{pmatrix}.$$

Let s_0, s_1, ..., s_{N-1} denote the standard involutions in G: s_i denote the matrix in G of the transposition $i \leftrightarrow i+1$, that is,

$$s_i = \begin{pmatrix} \mathrm{I}_{i-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & \mathrm{I}_{N-i-1} \end{pmatrix},$$

for every $i \in \{1, \dots, N-1\}$, and $s_0 = \Pi s_1 \Pi^{-1}$.

The finite Weyl group is $W_0 =< s_1, s_2, \dots, s_{N-1} >$, and the affine Weyl group is given by $W =< s_0, s_1, \dots, s_{N-1} >$.

We set

$$\mathcal{R}(g) = \Pi^{-1} g \Pi$$

with $g \in G$, so that $\mathcal{R}^N = 1$.

We will use repeatedly, and without further comment, the fact that induction commutes with conjugation: in particular conjugation by $\mathrm{Ad}\,\Pi^i$, $1 \le i \le N-1$. We will use this in the following form:

$$\mathcal{R}^{-1}(\mathrm{Ind}_{\mathcal{R}H}^{\mathcal{R}G}(\mathcal{R}\alpha)) \cong \mathrm{Ind}_H^G(\alpha). \tag{5}$$

Note that

$$\mathcal{R}(s_i) = s_{i+1}, \quad \text{with } i = 0,\, 1,\, \dots,\, N-1 \quad \mod N.$$

The extended affine Weyl group is given by $\widetilde{W} = W \rtimes < \Pi >$. We observe that

$$\widetilde{W} \cap \mathrm{GL}(N, \mathfrak{o}_F) = W_0. \tag{6}$$

The standard Iwahori subgroup is

$$I = \begin{pmatrix} \mathfrak{o}_F^\times & \mathfrak{o}_F & \cdots & \mathfrak{o}_F \\ \mathfrak{p}_F & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathfrak{o}_F \\ \mathfrak{p}_F & \cdots & \mathfrak{p}_F & \mathfrak{o}_F^\times \end{pmatrix}.$$

Let A be the apartment attached to the diagonal torus and let Δ denote the unique chamber of of A which is stabilized by $< \Pi > I$. We index the vertices L_0, L_1, ..., L_{N-1} of Δ in such a way that

- $s_i\Delta$ is the unique chamber of A which is adjacent to Δ and such that $s_i\Delta \cap \Delta$ is the $(N-2)$-simplex $\{L_0, \ldots, L_{N-1}\} \backslash \{L_i\}$;
- $\mathcal{R}(L_i) = L_{i+1}$ with $i = 0, 1, \ldots, N-1 \mod N$.

The L_i are the maximal standard parahoric subgroups of G,

$$L_i = I < s_0, s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{N-1} > I \ = \ \mathcal{R}^i(L_0),$$

and $L_0 = \mathrm{GL}(N, \mathfrak{o}_F)$.

The stabilizers of the facets of dimension $N-1$ of Δ are $J_0, J_1, \ldots, J_{N-1}$, where

$$J_i = I < s_i > I.$$

Each parahoric subgroup of G is defined by a facet of the building and the standard parahoric subgroups are the

$$J_S = I < s_j \ : \ j \in S > I,$$

where S is any subset of $\{0, 1, \ldots, N-1\}$ mod. N, [19, p. 118].

Hence, $I = J_\emptyset$, $J_i = J_{\{i\}}$, $L_i = J_{\{0,1,\ldots,i-1,i+1,\ldots,N-1\}}$.

The enlarged building $\beta^1 G$ is labellable, that is, there exists a simplicial map $\ell \colon \beta^1 G \to \Delta$, which preserves the dimensions of the simplices. The labelling is unique, up to the automorphisms of Δ. It allows us to fix an orientation of the simplices: one defines an incidence number $< \eta : \sigma >$ between an arbitrary facet $\eta = (\eta_0, \ldots, \eta_{i-1})$ of dimension i and any facet $\sigma = (\sigma_0, \ldots, \sigma_i)$ of dimension $i+1$ which contains η, as follows

$$< \eta : \sigma >= (-1)^i \ \text{ if } \{\ell(\eta_0), \ldots, \ell(\eta_{i-1})\} \backslash \{\ell(\sigma_0), \ldots, \ell(\sigma_i)\} = i.$$

The chamber homology groups are obtained by totalizing the bicomplex C_{**}

$$\begin{array}{ccccccccccccc} 0 & \longleftarrow & C_0 & \longleftarrow & \cdots & \longleftarrow & C_i & \longleftarrow & \cdots & \longleftarrow & C_{N-2} & \longleftarrow & C_{N-1} \\ & & \downarrow & & & & \downarrow & & & & \downarrow & & \downarrow \\ 0 & \longleftarrow & C_0 & \longleftarrow & \cdots & \longleftarrow & C_i & \longleftarrow & \cdots & \longleftarrow & C_{N-2} & \longleftarrow & C_{N-1} \end{array}$$

in which the chains are as follows:

$$C_i = \bigoplus_{\substack{S \subset \{0,1,\ldots,N-1\} \\ |S| = N-1-i}} R(J_S) \tag{7}$$

and each vertical map is given by $1 - \mathrm{Ad}\,\Pi$. In particular, we have

- $C_0 = R(L_0) \oplus R(L_1) \oplus \cdots \oplus R(L_{N-1})$,
- $C_{N-2} = R(J_0) \oplus R(J_1) \oplus \cdots \oplus R(J_{N-1})$,
- $C_{N-1} = R(I)$.

We will write an arbitrary element v in C_i as a $\binom{N}{i}$-uple $[\eta]$. Once an orientation has been chosen, the differentials are as follows: if $v \in C_i$ then

$$\partial(v) \;=\; \sum_{\substack{\eta\subset\sigma \\ \dim\eta = i}} (-1)^{\langle\eta:\sigma\rangle}\mathrm{Ind}_{G(\sigma)}^{G(\eta)}[\eta] \;\in C_{i-1}.$$

In particular:

- if $v \in C_{N-1}$ then $\partial(v) = (\mathrm{Ind}_I^{J_0}(v), \mathrm{Ind}_I^{J_1}(v), \ldots, \mathrm{Ind}_I^{J_{N-1}}(v))$,
- if $v \in C_0$ then $\partial(v) = 0$.

When $G = \mathrm{GL}(3)$, if $v = (v_0, v_1, v_2) \in C_1$ then $\partial(v)$ equals

$$(\mathrm{Ind}_{J_2}^{L_0}(v_2) - \mathrm{Ind}_{J_1}^{L_0}(v_1), \mathrm{Ind}_{J_0}^{L_1}(v_0) - \mathrm{Ind}_{J_2}^{L_1}(v_2), -\mathrm{Ind}_{J_0}^{L_2}(v_0) + \mathrm{Ind}_{J_1}^{L_2}(v_1)),$$

and, in the chain complex

$$0 \longleftarrow C_0 \longleftarrow C_1 \longleftarrow\!\!\longleftarrow C_2 \longleftarrow 0,$$

we have that v is a 1-cycle if and only if

$$\mathrm{Ind}_{J_2}^{L_0}(v_2) = \mathrm{Ind}_{J_1}^{L_0}(v_1), \mathrm{Ind}_{J_0}^{L_1}(v_0) = \mathrm{Ind}_{J_2}^{L_1}(v_2), \mathrm{Ind}_{J_0}^{L_2}(v_0) = \mathrm{Ind}_{J_1}^{L_2}(v_1),$$

i.e., if and only if the 1-chain (v_0, v_1, v_2) is *vertex compatible*. Note that a true representation in $R(I)$ can never be a 2-cycle; on the other hand, each 0-chain is a 0-cycle.

When we totalize the bicomplex we obtain the chain complex

$$0 \longleftarrow C_0 \longleftarrow C_0 \oplus C_1 \longleftarrow \cdots \longleftarrow C_{i-1} \oplus C_i \longleftarrow C_i \oplus C_{i+1} \longleftarrow \cdots \longleftarrow C_{N-1} \longleftarrow 0$$

Definition 2. The homology groups of this totalized complex are the chamber homology groups, as in [4].

To each point $\mathfrak{s} \in \mathfrak{B}(G)$ we will associate a sub-bicomplex $C_{**}(\mathfrak{s})$:

$$\begin{array}{ccccccccccc} 0 & \longleftarrow & C_0(\mathfrak{s}) & \longleftarrow & \cdots & \longleftarrow & C_i(\mathfrak{s}) & \longleftarrow & \cdots & \longleftarrow & C_{N-1}(\mathfrak{s}) \\ & & \downarrow & & & & \downarrow & & & & \downarrow \\ 0 & \longleftarrow & C_0(\mathfrak{s}) & \longleftarrow & \cdots & \longleftarrow & C_i(\mathfrak{s}) & \longleftarrow & \cdots & \longleftarrow & C_{N-1}(\mathfrak{s}) \end{array}$$

in which each vertical differential is 0. By an *invariant chain* we shall mean a chain invariant under $\mathrm{Ad}\,\Pi$.

Let $\mathfrak{s}$ be a point in $\mathfrak{B}(G)$ with $\mathfrak{s} = [M, \sigma]_G$. We recall that $W(M)$ denotes the group $\mathrm{N}_G(M)/M$. We take for M a standard Levi subgroup of G, isomorphic to $\mathrm{GL}(N_1) \times \cdots \times \mathrm{GL}(N_r)$, with $(N_1 \geq N_2 \geq \cdots \geq N_r)$ a partition of N.

Given a point $\mathfrak{s} \in \mathfrak{B}(G)$, fix an $\mathfrak{s}$-type (J, τ). Such types exist [8, 9, 10]. There exists a parahoric subgroup $J^{\mathfrak{s}}$ containing J such that $(J^{\mathfrak{s}}, \alpha := \mathrm{Ind}_J^{J^{\mathfrak{s}}} \tau)$ is also an $\mathfrak{s}$-type (see Theorems 4, 5, 6).

Then

- induce (if possible) each element in the orbit $W(M) \cdot \alpha$ to the standard parahoric subgroups containing $J^{\mathfrak{s}}$, and rotate, *i.e.,* apply $\mathcal{R}$, ..., $\mathcal{R}^{N-1}$,
- take the free abelian groups generated by all the irreducible components which arise in this way.

Each of our sub-complexes $C_{**}(\mathfrak{s})$ will come from some or all of this data. All the chain groups in $C_{**}(\mathfrak{s})$ are finitely generated free abelian groups and comprise invariant chains. The homology groups of the chain complex

$$0 \longleftarrow C_0(\mathfrak{s}) \longleftarrow C_1(\mathfrak{s}) \longleftarrow \cdots \longleftarrow C_{N-1}(\mathfrak{s}) \longleftarrow 0$$

will be denoted $h_*(\mathfrak{s})$. We call this the *little complex.*

When we totalize the associated bicomplex $C_{**}(\mathfrak{s})$ we obtain the chain complex

$$0 \longleftarrow C_0(\mathfrak{s}) \longleftarrow \cdots \longleftarrow C_{i-1}(\mathfrak{s}) \oplus C_i(\mathfrak{s}) \longleftarrow C_i(\mathfrak{s}) \oplus C_{i+1}(\mathfrak{s}) \longleftarrow \cdots \longleftarrow C_{N-1}(\mathfrak{s}) \longleftarrow 0$$

The following lemma will speed up our calculations.

Lemma 1. *The homology groups $H_*(\mathfrak{s})$ of this complex are given by*

$$H_0(\mathfrak{s}) = h_0(\mathfrak{s}), \quad H_N(\mathfrak{s}) = h_{N-1}(\mathfrak{s})$$

$$H_{i+1}(\mathfrak{s}) = h_i(\mathfrak{s}) \oplus h_{i+1}(\mathfrak{s}), \ \ 0 \leq i \leq N-2$$

$$H_{\mathrm{ev}}(\mathfrak{s}) = h_0(\mathfrak{s}) \oplus h_1(\mathfrak{s}) \oplus \cdots \oplus h_{N-1}(\mathfrak{s}) = H_{\mathrm{odd}}(\mathfrak{s})$$

The even (resp. odd) chamber homology is precisely the total homology of the little complex.

Proof. This is a direct consequence of the fact that each vertical differential in the bicomplex $C_{**}(\mathfrak{s})$ is 0. $\square$

4 Lattice chains and lattice sequences

Let V be an F-vector space of dimension N. We recall from [10, Def. 2.1] that a *lattice sequence* is a function Λ from $\mathbb{Z}$ to the set of $\mathfrak{o}_F$-lattices in V such that

- $i \geq j$ implies $\Lambda(i) \leq \Lambda(j)$;
- there exists $e = e(\Lambda) \in \mathbb{Z}$, $e \geq 1$, such that $\Lambda(i+e) = \mathfrak{p}_F \, \Lambda(i)$ for any $i \in \mathbb{Z}$.

The integer e is uniquely determined, and is called the *period* of Λ. We have $e \leq N$.

A lattice sequence which is injective as a function is called *strict*. We will put

$$\mathfrak{a}_n(\Lambda) := \{a \in A \,:\, a\Lambda(m) \subset \Lambda(m+n),\ m \in \mathbb{Z}\}\,, \quad n \in \mathbb{Z}. \tag{8}$$

The concept of lattice sequence generalizes the notion of lattice chain: as defined in [8, (1.11)], a *lattice chain* in V is a set $\mathcal{L} = \{L_i \,:\, i \in \mathbb{Z}\}$ of $\mathfrak{o}_F$-lattices L_i in V such that

- $L_i \supset L_{i+1}$, $L_i \neq L_{i+1}$, for any $i \in \mathbb{Z}$;
- there exists $e = e(\mathcal{L}) \in \mathbb{Z}$ such that $L_{i+e} = \mathfrak{p}_F \, L_i$, for any $i \in \mathbb{Z}$.

The integer e is uniquely determined, and is called the *period* of $\mathcal{L}$.

Let k_F denote the residue field of F. For each i, the quotient L_i/L_{i+1} is a k_F-vector space. Write

$$d_i = d_i(\mathcal{L}) := \dim_{k_F}(L_i/L_{i+1}). \tag{9}$$

The function $d(\mathcal{L})\colon i \mapsto d_i$, $i \in \mathbb{Z}$, is periodic of period dividing e, and we have

$$\sum_{i=0}^{e-1} d_i = N. \tag{10}$$

To each lattice chain $\mathcal{L}$ is attached a strict lattice sequence $\Lambda_{\mathcal{L}}$ defined by $\Lambda_{\mathcal{L}}(i) := L_i$, for $i \in \mathbb{Z}$. In the opposite direction, to each lattice sequence Λ is attached a lattice chain $\mathcal{L}_\Lambda$ defined by

$$\mathcal{L}_\Lambda := \{\Lambda(i) \,:\, i \in \mathbb{Z}\}\,. \tag{11}$$

As in [10, §2.6], we extend a lattice sequence Λ to a function on the real line $\mathbb{R}$ by setting

$$\Lambda(x) := \Lambda(\lceil x \rceil), \quad x \in \mathbb{R}, \tag{12}$$

where $\lceil x \rceil$ is the integer defined by the relation $\lceil x \rceil - 1 < x \leq \lceil x \rceil$.

Let Λ be a lattice sequence in V and let m be a positive integer. Then the function $m\Lambda$ from $\mathbb{Z}$ to the set of $\mathfrak{o}_F$-lattices in V defined by

$$(m\Lambda)(i) := \Lambda(i/m), \quad \text{for any } i \in \mathbb{Z},$$

is a lattice sequence in V with period $m\,e(\Lambda)$, and we have

$$(m\Lambda)(i) = \begin{cases} \Lambda(i/m) & \text{if } m \text{ divides } i, \\ \Lambda(1 + [i/m]) & \text{otherwise,} \end{cases} \tag{13}$$

and $(m\Lambda)(x) = \Lambda(x/m)$, for all $x \in \mathbb{R}$ (see [10, Prop. 2.7]).

If we have a lattice sequence Λ in V and an integer t, we can define a lattice chain $\Lambda + t$ by

$$(\Lambda + t)(i) := \Lambda(i + t), \;\; \text{for any } i \in \mathbb{Z}. \tag{14}$$

Let m be a positive integer, and let V^1, V^2, …, V^m be m finite-dimensional F-vector spaces. Let Λ^1, Λ^2, …, Λ^m be m lattices sequences in V, with periods e_1, e_2, …, e_m, respectively. We denote by $\Lambda = \Lambda^1 \oplus \cdots \oplus \Lambda^m$ the *direct sum* of Λ^1, …, Λ^m: we recall from [10, §2.8] that Λ is defined by

$$\Lambda(ex) = \Lambda^1(e_1 x) \oplus \cdots \oplus \Lambda^m(e_m x), \quad \text{for each } x \in \mathbb{R}, \text{ where } e = \operatorname{lcm}\{e_1, \ldots, e_m\}. \tag{15}$$

The following example occurs in the construction of [10, §7.2]. See also [10, Example 2.8].

Example 1. We assume given m lattice chains $\mathcal{L}^1$, $\mathcal{L}^2$, …, $\mathcal{L}^m$ in V^1, V^2, …, V^m, respectively, of same period e. We define a lattice chain

$$\mathcal{L} = \{L_i \, : \, i \in \mathbb{Z}\}$$

in V of period me by setting

$$L_{mj+k} := L_j^1 \oplus L_j^2 \oplus \cdots \oplus L_j^{m-k} \oplus L_{j+1}^{m-k+1} \oplus \cdots \oplus L_{j+1}^m,$$

any $j \in \mathbb{Z}$ and $0 \leq k \leq m-1$. Using (13), (14), we obtain

$$\Lambda_{\mathcal{L}} = (m\Lambda^1 - m + 1) \oplus \cdots \oplus (m\Lambda^{m-k} - k) \oplus \cdots \oplus (m\Lambda^{m-1} - 1) \oplus m\Lambda^m.$$

4.1 Addition of lattice chains

Let $A := \mathrm{End}_F(V)$ and let E/F be a subfield of A. We denote by $\mathfrak{o}_E$ the discrete valuation ring in E, by k_E its residue field, and by $e(E|F)$ the ramification degree of E/F.

Let $V^1, V^2, \ldots, V^m$ be m finite-dimensional F-vector spaces of dimensions $N_1, N_2, \ldots, N_m$, respectively. We assume that the field E preserves the spaces V^i. We may consider each V^l as a E-vector space of dimension $N_l/[E:F]$.

Let $\mathcal{L}^1, \mathcal{L}^2, \ldots, \mathcal{L}^m$ be m $\mathfrak{o}_E$-lattice chains in the E-vector spaces $V^1, V^2, \ldots, V^m$, respectively, of period $e'_1, e'_2, \ldots, e'_m$, respectively.

4.1.1 First addition procedure

We define first an $\mathfrak{o}_E$-lattice chain $\mathcal{L}^1 + \mathcal{L}^2 = \{L_j^{[1,2]} : j \in \mathbb{Z}\}$ in $V^1 \oplus V^2$ of period $e'_1 + e'_2$ by

$$L_i^{[1,2]} := \begin{cases} L_0^1 \oplus L_i^2, & \text{if } 0 \leq i \leq e'_2 - 1 \\ L^1_{i-e'_2} \oplus L^2_{e'_2}, & \text{if } e'_2 \leq i \leq e'_1 + e'_2 - 1. \end{cases} \tag{16}$$

Then let $\mathcal{L}^1 + \mathcal{L}^2 + \mathcal{L}^3 = \{L_i^{[1,3]} : i \in \mathbb{Z}\}$ be the $\mathfrak{o}_E$-lattice chain $(\mathcal{L}^1 + \mathcal{L}^2) + \mathcal{L}^3$ (which is the same as $\mathcal{L}^1 + (\mathcal{L}^2 + \mathcal{L}^3)$). By applying (16) to the two $\mathfrak{o}_E$-lattice chains $\mathcal{L}^1 + \mathcal{L}^2$ and $\mathcal{L}^3$, we get

$$L_i^{[1,3]} := \begin{cases} L_0^{[1,2]} \oplus L_i^3, & \text{if } 0 \leq i \leq e'_3 - 1 \\ L^{[1,2]}_{i-e'_3} \oplus L^2_{e'_3}, & \text{if } e'_3 \leq i \leq (e'_1 + e'_2) + e'_3 - 1, \end{cases}$$

that is, by using (16),

$$L_i^{[1,3]} := \begin{cases} L_0^1 \oplus L_2^0 \oplus L_i^3, & \text{if } 0 \leq i \leq e'_3 - 1 \\ L_0^1 \oplus L^2_{i-e'_3} \oplus L^2_{e'_3}, & \text{if } e'_3 \leq i \leq e'_2 + e'_3 - 1 \\ L^1_{i-e'_3-e'_2} \oplus L^2_{e'_2} \oplus L^2_{e'_3}, & \text{if } e'_2 + e'_3 \leq i \leq e'_1 + e'_2 + e'_3 - 1. \end{cases} \tag{17}$$

Using this procedure, we finally obtain an $\mathfrak{o}_E$-lattice chain

$$\mathcal{L}^1 + \cdots + \mathcal{L}^m = \mathcal{L}^{[1,m]} := \left\{ L_i^{[1,m]} : i \in \mathbb{Z} \right\} \tag{18}$$

of period $e'_1 + e'_2 + \cdots + e'_m$. We have

$$L_i^{[1,m]} = L_0^1 \oplus \cdots \oplus L_0^{j-1} \oplus L_k^j \oplus L^{j+1}_{e'_{j+1}} \oplus \cdots \oplus L^m_{e'_m}, \tag{19}$$

for $i = e'_{j+1} + \cdots + e'_m + k$ with $1 \leq j \leq m$ and $0 \leq k \leq e'_j - 1$.

We will need the $\mathfrak{o}_E$-lattice chain $\mathcal{L}^1 + \cdots + \mathcal{L}^m$ in the special case when $e'_1 = \cdots = e'_m = 1$. In that case, the equation (19) becomes

$$L_i^{[1,m]} = L_0^1 \oplus \cdots \oplus L_0^{m-i-1} \oplus L_0^{m-i} \oplus L_1^{m-i+1} \oplus \cdots \oplus L_1^m, \qquad (20)$$

for each $i \in \{0, 1, \ldots, m-1\}$.

Since the $\mathfrak{o}_E$-lattice chains $\mathcal{L}^1$, …, $\mathcal{L}^m$ all have period 1, we have $\mathfrak{p}_E^j L_0^l = L_j^l$, for each $l \in \{1, \ldots, m\}$ and each $j \in \mathbb{Z}$. Hence, since the $\mathfrak{o}_E$-lattice chain $\mathcal{L}^{[1,m]}$ is of period m, we have

$$L_{mj+k}^{[1,m]} = \mathfrak{p}_E^j L_k^{[1,m]} = L_j^1 \oplus \cdots \oplus L_j^{m-k-1} \oplus L_j^{m-k} \oplus L_{j+1}^{m-k+1} \oplus \cdots \oplus L_{j+1}^m, \quad (21)$$

for each $j \in \mathbb{Z}$ and each $k \in \{0, 1, \ldots, m-1\}$.

Then we have

$$L_{mj+k+1}^{[1,m]} = L_j^1 \oplus \cdots \oplus L_j^{m-k-1} \oplus L_{j+1}^{m-k} \oplus L_{j+1}^{m-k+1} \oplus \cdots \oplus L_{j+1}^m.$$

It follows that

$$L_{mj+k}^{[1,m]} / L_{mj+k+1}^{[1,m]} \cong L_j^{m-k} / L_{j+1}^{m-k},$$

for each $j \in \mathbb{Z}$ and each $k \in \{0, 1, \ldots, m-1\}$.

Hence, setting $d^l := d(\mathcal{L}^l)$ for any $1 \leq l \leq m$, we obtain

$$d_{mj+k}(\mathcal{L}^{[1,m]}) = d_j^{m-k} = d_0^{m-k} = \dim_E(V^{m-k}) = N_{m-k}/[E:F], \qquad (22)$$

by (10), since $e_{m-k} = 1$.

We may consider each $\mathcal{L}^l$, $1 \leq l \leq m$, as an $\mathfrak{o}_F$-lattice chain in the F-vector space V, of period $e(E|F)$ (see [8, (1.2.4)]). Then $\mathcal{L}^1 + \cdots + \mathcal{L}^m$, viewed as an $\mathfrak{o}_F$-lattice chain, has period $m\, e(E|F)$ (by [8, (1.2.4)]) and the equation (21) shows that it is the same as the chain $\mathcal{L}$ considered in the Example 1.

4.1.2 Second addition procedure

We keep assuming $e'_1 = \cdots = e'_m = 1$, and we will now consider the $\mathfrak{o}_E$-lattice chain

$$\mathcal{L}^m + \cdots + \mathcal{L}^1 = \left\{ L_i^{[m,1]} : i \in \mathbb{Z} \right\}.$$

We have

$$L_{mj+k}^{[m,1]} = L_{j+1}^1 \oplus \cdots \oplus L_{j+1}^k \oplus L_j^{k+1} \oplus \cdots \oplus L_j^m, \qquad (23)$$

for each $j \in \mathbb{Z}$ and each $k \in \{0, 1, \ldots, m-1\}$.

It gives

$$L_{mj+k}^{[m,1]} / L_{mj+k+1}^{[m,1]} \cong L_j^{k+1} / L_{j+1}^{k+1}, \qquad (24)$$

for each $j \in \mathbb{Z}$ and each $k \in \{0, 1, \dots, m-1\}$. Hence we obtain

$$d_{mj+k}(\mathcal{L}^{[m,1]}) = d_j^{k+1} = d_0^{k+1} = \dim_E(V^{k+1}) = N_{k+1}/[E:F], \qquad (25)$$

which in particular does not depend on m, in contrast with $d_{mj+k}(\mathcal{L}^{[1,m]})$.

As before, we may consider each $\mathcal{L}^l$, $1 \le l \le m$, as an $\mathfrak{o}_F$-lattice chain in the F-vector space V, of period $e(E|F)$. Then $\mathcal{L}^m + \dots + \mathcal{L}^1$, viewed as an $\mathfrak{o}_F$-lattice chain, has period $m\, e(E|F)$.

5 Hereditary $\mathfrak{o}_F$-orders

To any $\mathfrak{o}_F$-lattice chain $\mathcal{L} = \{L_i\}$ in V is attached the following sequence of $\mathfrak{o}_F$-lattices in A

$$\mathrm{End}^n_{\mathfrak{o}_F}(\mathcal{L}) := \{x \in A \,:\, xL_i \subset L_{i+n},\ i \in \mathbb{Z}\},$$

for each $n \in \mathbb{Z}$. In particular, $\mathfrak{A} = \mathfrak{A}(\mathcal{L}) := \mathrm{End}^0_{\mathfrak{o}_F}(\mathcal{L})$ is an hereditary $\mathfrak{o}_F$-order in A, and $\mathfrak{P} := \mathrm{End}^1_{\mathfrak{o}_F}(\mathcal{L})$ is the Jacobson radical of $\mathfrak{A}$. We will set

$$U(\mathfrak{A}) := \mathfrak{A}^\times \quad \text{and} \quad U^n(\mathfrak{A}) := 1 + \mathfrak{P}^n, \ \text{ for } n \ge 1. \qquad (26)$$

We put

$$\mathfrak{K}(\mathfrak{A}) := \left\{g \in \mathrm{Aut}_F(V) \,:\, g^{-1}\mathfrak{A}g = \mathfrak{A}\right\}. \qquad (27)$$

Definition 3. For any partition $(N_1, N_2, \dots, N_r)$ of N, we denote by

$$\mathfrak{A}(N_1, N_2, \dots, N_r)$$

the subset of $\mathrm{M}_N(F)$ consisting of the matrices of the following form: the (i,j)-block has dimension $N_i \times N_j$, $1 \le i, j \le r$, and its entries lie in $\mathfrak{o}_F$ if $i \le j$, in $\mathfrak{p}_F$ otherwise. Pictorially,

$$\mathfrak{A}(N_1, N_2, \dots, N_r) = \begin{pmatrix} \mathfrak{o}_F & \mathfrak{o}_F & \cdots & \mathfrak{o}_F \\ \mathfrak{p}_F & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathfrak{o}_F \\ \mathfrak{p}_F & \cdots & \mathfrak{p}_F & \mathfrak{o}_F \end{pmatrix}.$$

Let $e := e(\mathcal{L})$ and $d_i := d_i(\mathcal{L})$. For each $i \in \{0, 1, \dots, e-1\}$, we choose elements $v_{i,h} \in L_i$, $1 \le h \le d_i$ such that the cosets $v_{i,h} + L_{i+1}$ form a basis of the k_F-space L_i/L_{i+1}. Then

$$(v_{e-1,1}, v_{e-1,2}, \dots, v_{e-1,d_{e-1}}, v_{e-2,1}, v_{e-2,2}, \dots, v_{e-2,d_{e-2}}, \dots, v_{0,1}, v_{0,2}, \dots, v_{0,d_0})$$

is an F-basis of V. If we use this basis to identify A with the matrix algebra $\mathrm{M}_N(F)$, then $\mathfrak{A}$ becomes identified with $\mathfrak{A}(d_0, d_2, \ldots, d_{e-1})$.

Now let V^1, V^2, …, V^m be m finite-dimensional F-vector spaces as in sections 4.1.1, 4.1.2, and let $\mathcal{L}^1$, $\mathcal{L}^2$, …, $\mathcal{L}^m$ be m $\mathfrak{o}_E$-lattice chains in V^1, V^2, …, V^m, all of period 1. We put

$$\mathfrak{A}^{[m,1]} := \mathfrak{A}(\mathcal{L}^{[m,1]}),$$

where $\mathcal{L}^{[m,1]}$ is defined as in (23).

Let $(m_1, \ldots, m_r)$ be a partition of m. For each $i \in \{1, \ldots, r\}$, we set $\underline{m}_{\,i-1} := m_1 + \cdots + m_{i-1}$,

$$\mathcal{L}^{[\underline{m}_{\,i}, \underline{m}_{\,i-1}+1]} := \mathcal{L}^{\underline{m}_{\,i}} + \mathcal{L}^{\underline{m}_{\,i}-1} + \cdots + \mathcal{L}^{\underline{m}_{\,i-1}+2} + \mathcal{L}^{\underline{m}_{\,i-1}+1},$$

and

$$\mathfrak{A}^{[m_i, m_{i-1}+1]} := \mathfrak{A}(\mathcal{L}^{[\underline{m}_{\,i}, \underline{m}_{\,i-1}+1]}).$$

We set $m_0 := 0$. For each $i \in \{1, \ldots, r\}$, we define $V^{[m_{i-1}+1, m_i]}$ as

$$V^{[m_{i-1}+1, m_i]} := V^{\underline{m}_{\,i-1}+1} \oplus V^{\underline{m}_{\,i-1}+2} \oplus \cdots \oplus V^{\underline{m}_{\,i}}.$$

Let $M(m_1, \ldots, m_r)$ denote the stabilizer of the decomposition

$$V = \bigoplus_{i=1}^{r} V^{[m_{i-1}+1, m_i]}.$$

Lemma 2. *We have*

$$M(m_1, \ldots, m_r) \cap U(\mathfrak{A}^{[m,1]}) = \prod_{i=1}^{r} U(\mathfrak{A}^{[m_i, m_{i-1}]}).$$

Proof. We set $e := e(E|F)$. Let $l \in \{1, 2, \ldots, m\}$ and let $j \in \{0, 1, \ldots, e-1\}$. Since the $\mathfrak{o}_E$-lattice chain $\mathcal{L}^l$ has period 1, the equations (9) and (10) give

$$\dim_{k_E} L^l_j / L^l_{j+1} = \dim_{k_E} L^l_0 / L^l_1 = \frac{N_l}{[E:F]}.$$

It follows that

$$d^l_j = \dim_{k_F} L^l_j / L^l_{j+1} = [k_E : k_F] \dim_{k_E} L^l_j / L^l_{j+1} = [k_E : k_F] \frac{N_l}{[E:F]} = \frac{N_l}{e}.$$

Since $d^l_j = d^l_0 = N_l / e$, we may and do fix an $\mathfrak{o}_F$-basis $\mathcal{B}^l := (v^l_{0,1}, \ldots, v^l_{0,N_l})$ of $\mathcal{L}^l$, chosen to span L^l_0 over $\mathfrak{o}_F$. We put

$$v^l_{j,h} := \begin{cases} v^l_{0,h} & \text{if } 1 \le h \le \delta^l_j, \\ \varpi_F\, v^l_{0,h} & \text{if } \delta^l_j + 1 \le h \le N_l, \end{cases}$$

where

$$\delta_j^l := \dim_{k_F} L_j^l/L_e^l = (e-j)d_j^l = \frac{e-j}{e}\, N_l.$$

The $\mathfrak{o}_F$-lattice L_j^l is then the $\mathfrak{o}_F$-linear span of the set $\{v_{j,1}^l, \ldots, v_{j,N_l}^l\}$, the cosets $v_{j,h}^l + L_{j+1}^l$ ($1 \le h \le N_l/e$) form a basis of the k_F-space L_j^l/L_{j+1}^l, and

$$(v_{e-1,1}^l, \ldots, v_{e-1,N_l/e}^l, \ldots, v_{1,1}^l, \ldots, v_{1,N_l/e}^l, v_{0,1}^l, \ldots, v_{0,N_l/e}^l) = \mathcal{B}^l.$$

It follows that, for each $i \in \{1, 2, \ldots, r\}$,

$$\mathcal{B}^{[m_{i-1}+1, m_i]} := (\mathcal{B}^{\underline{m}_{i-1}+1}, \mathcal{B}^{\underline{m}_{i-1}+2}, \ldots, \mathcal{B}^{\underline{m}_i})$$

is an F-basis of the vector space $V^{[m_{i-1}+1, m_i]}$ such that the cosets

$$v_{j,h}^{k+1} + L_{mj+k+1}^{[m,1]}, \quad \text{for } 1 \le h \le N_{k+1}/e,$$

form a basis of the k_F-space

$$L_{mj+k}^{[m,1]}/L_{mj+k+1}^{[m,1]} \cong L_j^{k+1}/L_{j+1}^{k+1},$$

by (24).

Let $\mathcal{B}$ denote the F-basis of V defined as

$$\mathcal{B} := (\mathcal{B}^{[1,m_1]}, \mathcal{B}^{[m_1+1, m_1+m_2]}, \ldots, \mathcal{B}^{[m_{r-1}+1, m_r]}).$$

We observe that we have by construction

$$\mathcal{B} = \mathcal{B}^{[1,m]}, \tag{28}$$

where $\mathcal{B}^{[1,m]}$ is the F-basis corresponding to the partition m.

We will now use the basis $\mathcal{B}$ to identify $A = \mathrm{End}_F(V)$ with $\mathrm{M}_N(F)$ and use the basis $\mathcal{B}^{[\underline{m}_{i-1}+1, \underline{m}_i]}$ to identify $\mathrm{End}_F(V^{[m_{i-1}+1, m_i]})$ with $\mathrm{M}_{N(i)}(F)$, where

$$N(i) := N_{\underline{m}_{i-1}+1} + N_{\underline{m}_{i-1}+2} + \cdots + N_{\underline{m}_i}.$$

Then $\mathfrak{A}^{[m,1]}$ becomes identified with the matrices of the following form: the (h, h')-block has dimension

$$d_h(\mathcal{L}^{[m,1]}) \times d_{h'}(\mathcal{L}^{[m,1]}), \quad \text{if } 0 \le h, h' \le me - 1,$$

and its entries lie in $\mathfrak{o}_F$ if $i \le i'$, in $\mathfrak{p}_F$ otherwise.

Now the product $\prod_{i=1}^r \mathfrak{A}^{[\underline{m}_i, \underline{m}_{i-1}+1]}$ is viewed as diagonally embedded in $\mathrm{M}_N(F)$, and $\mathfrak{A}^{[\underline{m}_i, \underline{m}_{i-1}+1]}$ becomes then identified with the matrices of the following form: the $(\underline{m}_{i-1}e + j, \underline{m}_{i-1}e + j')$-block has dimension

$$d_j(\mathcal{L}^{[\underline{m}_i, \underline{m}_{i-1}+1]}) \times d_{j'}(\mathcal{L}^{[\underline{m}_i, \underline{m}_{i-1}+1]}), \quad \text{if } 0 \le j, j' \le m_i e - 1,$$

and its entries lie in $\mathfrak{o}_F$ if $j \le j'$, in $\mathfrak{p}_F$ otherwise. Then the result follows from (25). $\square$

6 Semisimple types

Let $G = \mathrm{GL}(N, F) = \mathrm{GL}(V)$ and let $\mathfrak{s} = [M, \sigma]_G$ be a point in the Bernstein spectrum $\mathfrak{B}(G)$. The Levi subgroup M is the stabilizer of a decomposition $V = \bigoplus_{l=1}^m V^l$ of V as a direct sum of nonzero subspaces V^l. We set $N_l := \dim_F V^l$, and $A_l := \mathrm{End}_F(V^l) \cong \mathrm{M}_{N_l}(F)$. Then $N_1 + \cdots + N_m = N$, and M is isomorphic to $\mathrm{GL}(N_1, F) \times \cdots \times \mathrm{GL}(N_m, F)$, and the supercuspidal representation σ of M is of the form $\sigma = \pi_1 \otimes \cdots \otimes \pi_m$, where π_l is an irreducible supercuspidal representation of the group $\mathrm{GL}(N_l, F)$, for $l = 1, \ldots, m$. We set $\mathfrak{t} := [M, \sigma]_M$.

By [8, Theorem (8.4.1)], for each l, there is a maximal simple type (J^l, λ^l) occuring in π_l. The pair $(J_M, \tau_M) := (J^1 \times \cdots \times J^m, \lambda^1 \otimes \cdots \otimes \lambda^m)$ is then an $\mathfrak{t}$-type in M.

By definition (see [8, (5.5.10)]), for each l, there exists an element $\beta_l \in A^l$ for which the algebra $E_l := F[\beta_l]$ is a field and a principal $\mathfrak{o}_F$-order $\mathfrak{A}^l$ in A^l, of period $e(E_l|F)$, with Jacobson radical $\mathfrak{P}_l$, such that

$$J^l = \begin{cases} J(\beta_l, \mathfrak{A}^l) & \text{(as defined in [8, (3.1.14)]) if } \beta_l \notin F, \\ U(\mathfrak{A}^l) & \text{if } \beta_l \in F. \end{cases}$$

For each $x \in A^l$, we will write

$$\nu_{\mathfrak{A}^l}(x) := \max\{n \in \mathbb{Z} \,:\, x \in \mathfrak{P}_l\}. \tag{29}$$

Let $\mathcal{L}^l$ denote the $\mathfrak{o}_E$-lattice chain defining the maximal $\mathfrak{o}_E$-order $\mathfrak{B}^l := \mathfrak{A}^l \cap \mathrm{End}_E(V^l)$. We have

$$J(\beta, \mathfrak{A}^l)/J^1(\beta, \mathfrak{A}^l) = U(\mathfrak{B}^l)/U^1(\mathfrak{B}^l) \cong \mathrm{GL}(f_l, k_E). \tag{30}$$

6.1 Simple types

We assume in this subsection that the N_l are all equal to N/m and that $\pi_l \cong \pi_j \chi_j$, with χ_j an unramified character of $\mathrm{GL}(N/m, F)$, for each $l, j \in \{1, \ldots, m\}$. In particular, M is then isomorphic to $\mathrm{GL}(N/m, F)^m$, and by [8, Theorem (8.4.2)], we can assume that all the β_l, all the $\mathfrak{A}^l$, all the $\mathcal{L}^l$, all the J^l and all the λ^l are equal. We will denote by E (resp. β) the common value of the E_l (resp. β_l).

Using the second addition procedure 4.1.2, we define: the $\mathfrak{o}_E$-lattice chain

$$\mathcal{L} := \mathcal{L}^m + \mathcal{L}^{m-1} + \cdots + \mathcal{L}^1, \quad \text{and} \quad \mathfrak{A} := \mathrm{End}^0_{\mathfrak{o}_F}(\mathcal{L}). \tag{31}$$

If β belongs to F, we set $J := U(\mathfrak{A})$. Otherwise, let $n := -\nu_{\mathfrak{A}^1}(\beta)$, then $[\mathfrak{A}, mn, 0, \beta]$ is a simple stratum in the sense of [8, Definition (1.5.5)], let $(J, \lambda) := (J(\beta, \mathfrak{A}), \lambda)$ be the corresponding simple type in G.

Let $\mathfrak{B}$ denote the principal $\mathfrak{o}_E$-order in $B := \mathrm{M}_{N/[E:F]}(E)$ defined by $\mathfrak{B} := B \cap \mathfrak{A}$. We have $m = e(\mathfrak{B}) = e(\mathfrak{B}|\mathfrak{o}_E)$. In the case when $\beta \in F$, we have $m = e(\mathfrak{A})$.

Definition 4. We set

$$\mathfrak{A}^{\mathfrak{s}} := \mathfrak{A}(N/m, \dots, N/m) \quad \text{and} \quad J^{\mathfrak{s}} := U(\mathfrak{A}^{\mathfrak{s}}),$$

where $\mathfrak{A}(N/m, \dots, N/m)$ is defined by Definition 3.

Lemma 3. *The $\mathfrak{o}_F$-order $\mathfrak{A}$ is contained in the $\mathfrak{o}_F$-order $\mathfrak{A}^{\mathfrak{s}}$.*

Proof. We have $\mathfrak{A} = \mathfrak{A}(N/e(\mathfrak{A}), \dots, N/e(\mathfrak{A}))$. In the case when $J = U(\mathfrak{A})$, we have $\mathfrak{A}^{\mathfrak{s}} = \mathfrak{A}$. Otherwise, the statement follows immediately from the above descriptions of the orders $\mathfrak{A}$, $\mathfrak{A}^{\mathfrak{s}}$, and from the fact (see [8, Proposition (1.2.4)]) that

$$e(\mathfrak{A}) = m \cdot e(E|F).$$

Indeed, from the above descriptions of the orders $\mathfrak{A}$, $\mathfrak{A}^{\mathfrak{s}}$, we have

$$\mathfrak{A}^{\mathfrak{s}} \cap U = \mathfrak{A} \cap U, \;\; \mathfrak{A}^{\mathfrak{s}} \cap \overline{U} = \mathfrak{A} \cap \overline{U}, \tag{32}$$

$$M \cap \mathfrak{A}^{\mathfrak{s}} \;\cong\; (\mathrm{GL}(N/m, \mathfrak{o}_F))^m, \tag{33}$$

while $M \cap \mathfrak{A}$ is isomorphic to the product of m copies of the order of $e(E|F) \times e(E|F)$ blocks matrices of the following form: the (j,l)-block has dimension $N/e(\mathfrak{A}) \times N/e(\mathfrak{A}) = (N/e(E|F)m \times N/e(E|F)m)$, $0 \le j, l \le e(E|F) - 1$, and its entries lie in $\mathfrak{o}_F$ if $j \le l$, in $\varpi_F \mathfrak{o}_F$ otherwise, so that $M \cap \mathfrak{A} \subset M \cap \mathfrak{A}^{\mathfrak{s}}$. $\square$

We set

$$f = \frac{N}{[E:F] \cdot m}. \tag{34}$$

Let K/E be an unramified field extension of degree f with

$$K^{\times} \subset \mathfrak{K}(\widetilde{\mathfrak{B}}),$$

where $\mathfrak{K}(\widetilde{\mathfrak{B}})$ is defined by (27), and let $C = \mathrm{End}_K(V) \cong \mathrm{M}_m(K)$. We view ϖ_E as a prime element of K. For $i = 1, \dots, m-1$, let $s_{i,C}$ denote the matrix in C of the transposition $i \leftrightarrow i+1$, that is,

$$s_{i,C} = \begin{pmatrix} \mathrm{I}_{i-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & \mathrm{I}_{m-i-1} \end{pmatrix},$$

and let $s_{0,C} = \Pi_{m,C} s_{1,C} \Pi_{m,C}^{-1}$, with

$$\Pi_{m,C} = \begin{pmatrix} 0 & \mathrm{I}_{m-1} \\ \varpi_E & 0 \end{pmatrix}.$$

We fix the embedding

$$\bigotimes \mathrm{I}_{N/m} \colon C \hookrightarrow \mathrm{M}_N(K) \quad c = (c_{ij}) \mapsto c \otimes \mathrm{I}_{N/m} = (c_{ij}\mathrm{I}_{N/m}),$$

$c \otimes \mathrm{I}_{N/m}$ being a block matrix with scalar blocks.

Let $\widetilde{W}_C$ be the group generated by

$$S = \{s_{0,C} \otimes \mathrm{I}_{N/m}, s_{1,C} \otimes \mathrm{I}_{N/m}, \ldots, s_{m-1,C} \otimes \mathrm{I}_{N/m}\}.$$

Then $(\widetilde{W}_C, S)$ is a Coxeter group of type $\tilde{A}_{m-1}$.

Theorem 4. *The representation $\alpha = \mathrm{Ind}_J^{J^{\mathfrak{s}}}(\lambda)$ is irreducible. Hence the pair $(J^{\mathfrak{s}}, \alpha)$ is an $\mathfrak{s}$-type.*

Proof. In the case when $J = U(\mathfrak{A})$, we have $J^{\mathfrak{s}} = J$, so the result follows trivially in this case. We will assume from now on that $J = J(\beta, \mathfrak{A})$. For any $i \in \{1, \ldots, m-1\}$,

$$s_{i,C} \otimes \mathrm{I}_{N/m} = \begin{pmatrix} \mathrm{I}_{(i-1)N/m} & & & \\ & 0 & \mathrm{I}_{N/m} & \\ & \mathrm{I}_{N/m} & 0 & \\ & & & \mathrm{I}_{(m-i-1)N/m} \end{pmatrix} \notin J^{\mathfrak{s}},$$

and

$$\Pi_{m,C} \otimes \mathrm{I}_{N/m} = \begin{pmatrix} 0 & \mathrm{I}_{(m-1)N/m} \\ \varpi_E \mathrm{I}_{N/m} & 0 \end{pmatrix} \notin J^{\mathfrak{s}}).$$

Hence $\widetilde{W}_C \cap J^{\mathfrak{s}} = \{1\}$, which gives

$$J^{\mathfrak{s}} \cap (J \cdot \widetilde{W}_C \cdot J) = J. \tag{35}$$

Then the result follows from the fact (see [8, Propositions (5.5.11) and (5.5.14) (iii)]) that

$$I_G(\lambda) \subset J \cdot \widetilde{W}_C \cdot J.$$

$\square$

6.2 In the Levi subgroup $\widetilde{M}$

We will now consider the case of an arbitrary point $\mathfrak{s} = [M, \sigma]_G$ in $\mathfrak{B}(G)$, with $G = \mathrm{GL}(N, F)$. Let $\widetilde{M}$ denote the unique Levi subgroup of G which contains $\mathrm{N}_{\mathfrak{t}}$ (see 4) and is minimal for this property.

We write $\sigma = \pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_m$ as

$$\sigma = (\sigma_1, \ldots, \sigma_1, \sigma_2, \ldots, \sigma_2, \ldots, \sigma_t, \ldots, \sigma_t),$$

where σ_j, a supercuspidal representation of $\mathrm{GL}(N'_j, F)$, is repeated ε_j times, $1 \le j \le t$, and σ_1, ..., σ_t are pairwise distinct (after unramified twist). The integers ε_1, ε_2, ..., ε_t are called the *exponents* of σ. Then we have

$$M \;\cong\; \mathrm{GL}(N'_1, F)^{\varepsilon_1} \times \mathrm{GL}(N'_2, F)^{\varepsilon_2} \times \cdots \times \mathrm{GL}(N'_t, F)^{\varepsilon_t},$$

and

$$\widetilde{M} \;\cong\; \mathrm{GL}(\varepsilon_1 N'_1, F) \times \mathrm{GL}(\varepsilon_2 N'_2, F) \times \cdots \times \mathrm{GL}(\varepsilon_t N'_t, F).$$

For every $j \in \{1, \ldots, t\}$, we set

$$\mathfrak{s}_j = [\mathrm{GL}(N'_j, F)^{\varepsilon_j}, \sigma_j^{\otimes \varepsilon_j}]_{\mathrm{GL}(\varepsilon_j N'_j, F)}.$$

Then let (K^j, τ^j) be the $\mathfrak{s}_j$-type in $\mathrm{GL}(\varepsilon_j N'_j, F)$ (a simple type) defined as in the previous section, and let $(\widetilde{K}^j, \widetilde{\tau}^j)$ be the "modified simple type" attached to (K^j, τ^j) as in [10, proof of Prop. 1.4].

Lemma 4. *We have $\widetilde{K}^j \subset J^{\mathfrak{s}_j}$ and $\alpha_i = \mathrm{Ind}_{\widetilde{K}^j}^{J^{\mathfrak{s}_j}}(\widetilde{\tau}^j)$ is irreducible.*

Proof. There is an isomorphism of Hecke algebras

$$\mathcal{H}(\mathrm{GL}(\varepsilon_j N'_j), \widetilde{\tau}^j) \;\cong\; \mathcal{H}(\mathrm{GL}(\varepsilon_j N_j), \tau^j)$$

such that, if $\tilde{f} \in \mathcal{H}(\mathrm{GL}(\varepsilon_j N'_j), \widetilde{\tau}^j)$ has support $\widetilde{K}^j g \widetilde{K}^j$, for some element $g \in \mathrm{GL}(\varepsilon_j N'_j, F)$, then its image f in $\mathcal{H}(\mathrm{GL}(\varepsilon_j N'_j), \tau^j)$ has support $K^j g K^j$ (see [8, (7.2.19)]). Then the result follows from Theorem 4. $\square$

We set

$$\mathfrak{s}_{\widetilde{M}} = [M, \sigma]_{\widetilde{M}}, \quad \widetilde{\mathfrak{A}}^{\mathfrak{s}} = \mathfrak{A}^{\mathfrak{s}_1} \times \cdots \times \mathfrak{A}^{\mathfrak{s}_t}, \quad \widetilde{J}^{\mathfrak{s}} = U(\widetilde{\mathfrak{A}}^{\mathfrak{s}}),$$

$$\widetilde{K} = \widetilde{K}^1 \times \cdots \times \widetilde{K}^t \subset \widetilde{M}, \quad \widetilde{\tau} = \widetilde{\tau}^1 \otimes \cdots \otimes \widetilde{\tau}^t. \tag{36}$$

Note that

$$\widetilde{J}^{\mathfrak{s}} = \widetilde{M} \cap J^{\mathfrak{s}}. \tag{37}$$

It immediately follows from Lemma 4 that:

Lemma 5. *We have $\widetilde{K} \subset \widetilde{J}^{\mathfrak{s}}$ and $\widetilde{\alpha} = \mathrm{Ind}_{\widetilde{K}}^{\widetilde{J}^{\mathfrak{s}}}(\widetilde{\tau})$ is irreducible.*

6.3 Review of endo-classes

We recall that a *simple pair* (k, β) *over* F consists of an integer k and a nonzero element β generating a field extension E of F such that

$$-k > \max\left\{k_0(\beta, \mathfrak{A}(E)), \nu_E(\beta)\right\},$$

where ν_E is the standard additive valuation on E and $k_0(\beta, \mathfrak{A}(E))$ is defined by [8, (1.4.5)], with $\mathfrak{A}(E)$ denoting the unique hereditary $\mathfrak{o}_F$-order in $\mathrm{End}_F(E)$ such that $\mathfrak{K}(\mathfrak{A}(E)) \supset E^\times$.

Let (k, β) be a given simple pair in which $k \geq 0$. A *ps-character* (attached to the simple pair (k, β)) is then a triple (Θ, k, β), where Θ is a simple-character-valued function, such that to each triple $(V, \mathfrak{B}, m)$, where V is a finite-dimensional E-vector space, $\mathfrak{B}$ is a hereditary $\mathfrak{o}_E$-order in $\mathrm{End}_E(V)$, and m is an integer such that $[m/e(\mathfrak{B}|\mathfrak{o}_E)] = k$, the function Θ attaches a simple character $\Theta(\mathfrak{A}) \in \mathcal{C}(\mathfrak{A}, m, \beta)$, called the *realization of* Θ *on* $\mathfrak{A}$ *of order* m. (If we put $n := -\nu_E(\beta)\, e(\mathfrak{B})$, the stratum $[\mathfrak{A}, n, m, \beta]$ is simple and the simple character set $\mathcal{C}(\mathfrak{A}, m, \beta)$ of [8, (3.2)] is defined.)

These realizations are subject to the following coherence condition: if we have two realizations $\Theta(\mathfrak{A}_1)$ and $\Theta(\mathfrak{A}_2)$ of on orders $\mathfrak{A}_1$, $\mathfrak{A}_2$, they are related by $\Theta(\mathfrak{A}_2) = \tau_{\mathfrak{A}_1,\mathfrak{A}_2,\beta}(\Theta(\mathfrak{A}_1))$, where

$$\tau_{\mathfrak{A}_1,\mathfrak{A}_2,\beta} \colon \mathcal{C}(\mathfrak{A}_1, m, \beta) \to \mathcal{C}(\mathfrak{A}_2, m, \beta)$$

is the canonical bijection of [8, (3.6.14)].

Following [10, §4.3], we will say that two ps-characters (Θ_1, k_1, β_1) and (Θ_2, k_2, β_2) are *endo-equivalent* if there exists an F-vector space V, hereditary $\mathfrak{o}_F$-orders $\mathfrak{A}_1$, $\mathfrak{A}_2$ in $\mathrm{End}_F(V)$, and realizations $\Theta_i(\mathfrak{A}_i)$ of the Θ_i of same level, such that $\mathfrak{A}_1 \cong \mathfrak{A}_2$ as $\mathfrak{o}_F$-orders, and such that the simple characters $\Theta_i(\mathfrak{A}_i)$ intertwine in $\mathrm{Aut}_F(V)$. Endo-equivalence in equivalence relation on the set of ps-characters over F. One refers to the equivalence classes as *endo-classes* of simple characters.

If the supercuspidal representation π_l of $\mathrm{GL}(N_l, F)$ contains the trivial character of $U^1(\mathfrak{A}^l) = 1 + \mathfrak{P}_l$, then π_l is said to be of level-zero. Otherwise, there exists a simple stratum $[\mathfrak{A}^l, n_l, 0, \beta_l]$ in A_l and a simple character $\theta_l \in \mathcal{C}(\mathfrak{A}^l, 0, \beta_l)$ such that the restriction of λ^l to $H^1(\beta_l, \mathfrak{A}^l)$ is a multiple of θ_l. (Here $H^1(\beta_l, \mathfrak{A}^l)$ is defined as in [8, (3.1.14)].) Since $[\mathfrak{A}^l, n_l, 0, \beta_l]$ is simple, we have $n_l = -\nu_{\mathfrak{A}^l}(\beta_l)$. Then each representation λ^l is given as follows. There is a unique irreducible representation η_l of $J^1(\beta, \mathfrak{A}^l)$ whose restriction to $H^1(\beta, \mathfrak{A}^l)$ is a multiple of θ_l. The representation η_l extends to a representation κ_l which is a β-extension of η_l, and we have $\lambda^l = \kappa_l \otimes \rho_l$, where ρ_l is the inflation of an irreducible representation of $\mathrm{GL}(f_l, k_E)$, with f_l defined by (30).

If the representation π_l is of level zero, we set $\Theta_{\pi_l} = \{\Theta^0\}$, where Θ^0 is the trivial ps-character (that is, if $\mathfrak{A}$ is a hereditary $\mathfrak{o}_F$-order in some $\mathrm{End}_F(V)$, the realization of Θ^0 on $\mathfrak{A}$ is the trivial character of $U^1(\mathfrak{A})$). Otherwise, the simple character θ_i determines a ps-character $(\Theta_l, 0, \beta)$ and hence an endo-class Θ_{π_l}.

We will denote by $\Theta(1)$, $\Theta(2)$, …, $\Theta(q)$ the distinct endo-classes arising in the set $\{\Theta_{\pi_1}, \dots, \Theta_{\pi_m}\}$.

6.4 The homogeneous case

In this subsection, we assume that all the representations π_1, π_2, …, π_m admit the same endo-class. It follows that all the elements β_1, …, β_m may be assumed to be equal. We will denote by E (resp. β) the common value of the E_l (resp. β_l).

Let $l \in \{1, \dots, m\}$, and let $(v_1^l, v_2^l, \dots, v_{N_l}^l)$ be an F-basis of V^l, with respect to which $\mathfrak{A}^l = \mathfrak{A}(\mathcal{L}^l)$ is identified with $\mathfrak{A}(N_l/e(E|F), \dots, N_l/e(E|F))$. We have $L_0^l = \mathfrak{o}_F v_1^l \oplus \cdots \oplus \mathfrak{o}_F v_{N_l}^l$. We set

$$L_{i,\max}^l := \mathfrak{p}^i L_0^l, \quad \text{for any } i \in \mathbb{Z}.$$

Then

$$\mathcal{L}_{\max}^l := \left\{ L_{i,\max}^l \, : \, i \in \mathbb{Z} \right\} \tag{38}$$

is an $\mathfrak{o}_F$-lattice chain in V^l of period 1, and we have

$$\mathfrak{A}(\mathcal{L}_{\max}^l) := \mathrm{End}_{\mathfrak{o}_F}^0(\mathcal{L}_{\max}^l) = \mathfrak{A}(N_l) = \mathrm{M}_{N_l}(\mathfrak{o}_F) \supset \mathfrak{A}^l.$$

Following the second addition procedure defined in the subsection 4.1, we assemble the $\mathfrak{o}_E$-lattices chains $\mathcal{L}^1$, …, $\mathcal{L}^m$ into the $\mathfrak{o}_E$-lattice chain

$$\bar{\mathcal{L}} := \mathcal{L}^m + \mathcal{L}^{m-1} + \cdots + \mathcal{L}^1 \tag{39}$$

in V, of period m, and we assemble the $\mathfrak{o}_F$-lattices chains $\mathcal{L}_{\max}^1$, …, $\mathcal{L}_{\max}^m$ into the $\mathfrak{o}_F$-lattice chain

$$\bar{\mathcal{L}}_{\max} := \mathcal{L}_{\max}^m + \mathcal{L}_{\max}^{m-1} + \cdots + \mathcal{L}_{\max}^1 \; = \; \left\{ \bar{L}_{\max,i} \, : \, i \in \mathbb{Z} \right\} \tag{40}$$

in V, of period m. Let $j \in \mathbb{Z}$ and $k \in \{0, 1, \dots, m-1\}$. From (24), we have

$$\bar{L}_{\max,mj+k} / \bar{L}_{\max,mj+k+1} \; \cong \; L_{\max,j}^{k+1} / L_{\max,j+1}^{k+1}.$$

Hence:

$$d_{mj+k}(\bar{\mathcal{L}}_{\max}) = N_{k+1}. \tag{41}$$

It follows that

$$\mathfrak{A}(\bar{\mathcal{L}}_{\max}) = \mathfrak{A}(N_1, N_2, \ldots, N_m).$$

We put

$$B := \mathrm{End}_E(V) \quad \text{and} \quad \mathfrak{B} := \mathrm{End}^0_{\mathfrak{o}_E}(\bar{\mathcal{L}}). \tag{42}$$

Considering $\bar{\mathcal{L}}$ as an $\mathfrak{o}_F$-lattice chain, we put

$$\mathfrak{A} := \mathrm{End}^0_{\mathfrak{o}_F}(\bar{\mathcal{L}}). \tag{43}$$

We have $\mathfrak{B} = \mathfrak{A} \cap B$.

The following definition, lemma and theorem generalize Definition 4, Lemma 3, and Theorem 4, respectively.

Definition 5. We set

$$\mathfrak{A}^{\mathfrak{s}} := \mathfrak{A}(N_1, N_2, \ldots, N_m) \quad \text{and} \quad J^{\mathfrak{s}} := U(\mathfrak{A}^{\mathfrak{s}}).$$

Lemma 6. *The $\mathfrak{o}_F$-order $\mathfrak{A}$ is contained in the $\mathfrak{o}_F$-order $\mathfrak{A}^{\mathfrak{s}}$.*

Proof. We have

$$\mathfrak{A}^{\mathfrak{s}} \cap U = \mathfrak{A} \cap U, \;\; \mathfrak{A}^{\mathfrak{s}} \cap \overline{U} = \mathfrak{A} \cap \overline{U},$$

$$M \cap \mathfrak{A}^{\mathfrak{s}} \cong \prod_{l=1}^{m} \mathrm{GL}(N_l, \mathfrak{o}_F).$$

In the notation of subsection 4.1.2, setting $(m_1, \ldots, m_r) = (1, \ldots, 1)$, we have $r = m$, $M = M(m_1, \cdots, m_r)$. Then $\underline{m}_{\,l-1} = l - 1 = \underline{m}_{\,l} - 1$, hence $\bar{\mathcal{L}}^{[\underline{m}_{\,l}, \underline{m}_{\,l-1}+1]} = \mathcal{L}^l$,

$$\mathfrak{A}^l = \mathfrak{A}(\mathcal{L}^l) \cong \mathfrak{A}(N_l/e(E|F), \ldots, N_l/e(E|F)),$$

and Lemma 2 gives

$$M \cap \mathfrak{A} \cong \prod_{l=1}^{m} U(\mathfrak{A}^l).$$

Since $U(\mathfrak{A}^l) \subset \mathrm{GL}(N_l, \mathfrak{o}_F)$, the result follows. $\square$

We set

$$n := \max(n_1, \ldots, n_m). \tag{44}$$

Lemma 7. *With notation as above, $[\mathfrak{A}, nm, 0, \beta]$ is a simple stratum.*

Proof. We have to check that the four conditions occuring in [8, Definition (1.5.5)] are satisfied.

(i) We know that the algebra $E = F[\beta]$ is a field, since the strata $[\mathfrak{A}^l, n_l, 0, \beta]$ are simple.

(ii) We defined $\bar{\mathcal{L}} = \{\bar{L}_i : i \in \mathbb{Z}\}$ to be an $\mathfrak{o}_E$-lattice chain in the E-vector space V. Hence, by [8, Proposition (1.2.1)], we have $E^\times \subset \mathfrak{K}(\mathfrak{A})$.

(iii) Let $l \in \{1, \ldots, m\}$. We set $\mathfrak{Q}_l := \mathfrak{B}_l \cap \mathfrak{P}_l$. Since $\nu_{\mathfrak{A}^l}(\beta) = -n_l$, the definition (29) for $\nu_{\mathfrak{A}^l}$ shows that

$$\beta \in \operatorname{End}_E(V^l) \cap \mathfrak{P}_l^{-n_l} \quad \text{and} \quad \beta \notin \mathfrak{P}_l^{-n_l+1},$$

that is,

$$\beta \in \mathfrak{Q}_l^{-n_l} \quad \text{and} \quad \beta \notin \mathfrak{Q}_l^{-n_l+1}.$$

By [8, Proposition (1.2.4)], we know that $\mathfrak{Q}_l$ is the Jacobson radical of the $\mathfrak{o}_E$-order $\mathfrak{B}_l$. Hence $\mathfrak{Q}_l^i = \operatorname{End}^i_{\mathfrak{o}_E}(\mathcal{L}^l)$ for each $i \in \mathbb{Z}$, and $\beta(L_j^l)$ is contained in $L^l_{j-n_l}$ and not in $L^l_{j-n_l+1}$. Now, it follows from (23) that

$$\beta(L_{mj+k}) = \beta(L^1_{j+1}) \oplus \cdots \oplus \beta(L^k_{j+1}) \oplus \beta(L^{k+1}_j) \oplus \cdots \oplus \beta(L^m_j),$$

for each $j \in \mathbb{Z}$ and each $k \in \{0, 1, \ldots, m-1\}$. Since $n = \max(n_1, \ldots, n_m)$, we have $L^l_{j-n_l} \subset L^l_{j-n}$, for each l. It gives $\beta(L_{mj+k}) \subset L_{m(j-n)+k}$. On the other side there exists $l_0 \in \{1, \ldots, m\}$ such that $n = n_{l_0}$, and hence $\beta(L_j^{l_0})$ is not contained in L^l_{j-n+1}. It follows that $\beta(L_{mj+k})$ is not contained in $L_{m(j-n)+k+1}$, that is,

$$\beta \in \mathfrak{Q}^{-n} \quad \text{and} \quad \beta \notin \mathfrak{Q}^{-n+1},$$

where $\mathfrak{Q}$ denotes the Jacobson radical of $\mathfrak{B}$. Since $\mathfrak{Q}^i = B \cap \mathfrak{P}^i$ for each $i \in \mathbb{Z}$ (by [8, Proposition (1.2.4)]), we get $\nu_{\mathfrak{A}}(\beta) = -nm$.

(iv) Let $A(E) := \operatorname{End}_F(E)$. The algebra $A(E)$ contains the principal $\mathfrak{o}_F$-order

$$\mathfrak{A}(E) := \operatorname{End}^0_{\mathfrak{o}_F}\left(\left\{\mathfrak{p}_E^i : i \in \mathbb{Z}\right\}\right).$$

We have $E^\times \subset \mathfrak{K}(\mathfrak{A}(E))$ and [8, Proposition (1.4.13) (ii)] gives

$$k_0(\beta, \mathfrak{A}) = mk_0(\beta, \mathfrak{A}(E)) = k_0(\beta, \mathfrak{A}^l), \quad \text{for each } l \in \{1, \ldots, m\}.$$

Since $[\mathfrak{A}^l, n_l, 0, \beta]$ is a simple stratum, we have

$$0 < -k_0(\beta, \mathfrak{A}^l).$$

Hence $0 < -k_0(\beta, \mathfrak{A})$ and $[\mathfrak{A}, mn, 0, \beta]$ is simple.

□

Since $[\mathfrak{A}, nm, 0, \beta]$ is a simple stratum, we can associate to it the compact open subgroups $J(\beta, \mathfrak{A})$ and $H^1(\beta, \mathfrak{A})$ of $U(\mathfrak{A})$, defined following [8, (3.1.14)].

As in [8, §7.1, 7.2], the set

$$K := H^1(\beta, \mathfrak{A}) \cap \overline{U} \cdot J(\beta, \mathfrak{A}) \cap P \tag{45}$$

is then a subgroup of $U(\mathfrak{A})$ containing $H^1(\beta, \mathfrak{A})$.

Definition 5 and Lemma 6 imply

$$K \subset J^{\mathfrak{s}}. \tag{46}$$

As in [10, §7.2.1], it admits an irreducible representation κ, trivial on $K \cap \overline{U}$, $K \cap U$, whose restriction to $H^1(\beta, \mathfrak{A})$ is a multiple of $\theta = \Theta(\mathfrak{A})$, and such that $\kappa_{|K\cap M}$ is of the form $\kappa'_1 \otimes \cdots \otimes \kappa'_m$ for some β-extension κ'_l of η_l. As in [10, §7.2], we can choose the decomposition $\lambda^l = \kappa_l \otimes \rho_l$ above so that $\kappa_l = \kappa'_l$; we assume this has been done. We have canonically

$$K/K \cap J^1(\beta, \mathfrak{A}) \cong \prod_{l=1}^{m} J(\beta, \mathfrak{A}^l)/J^1(\beta, \mathfrak{A}^l) \cong \prod_{l=1}^{m} \mathrm{GL}(f_l, k_E),$$

and we can inflate the cuspidal representation $\rho_1 \otimes \cdots \otimes \rho_m$ of $\prod_{l=1}^m \mathrm{GL}(f_l, k_E)$ to a representation ρ of K and form

$$\tau = \kappa \otimes \rho. \tag{47}$$

Moreover similar proofs of those of [10, Theorem 7.2.1, Main Theorem 8.2] show that (K, τ) is a G-cover of the pair $(\widetilde{K}, \widetilde{\tau})$ defined in (36) and give the following formula for the intertwining:

$$\mathcal{I}_G(\tau) = K \cdot \mathcal{I}_{\widetilde{M}}(\widetilde{\tau}) \cdot K. \tag{48}$$

Theorem 5. *Let $J^{\mathfrak{s}}$ be as in Definition 5. Then the representation $\alpha := \mathrm{Ind}_K^{J^{\mathfrak{s}}}(\tau)$ is irreducible. Hence the pair $(J^{\mathfrak{s}}, \alpha)$ is an $\mathfrak{s}$-type.*

Proof. Using equations (46) and (48), we obtain

$$\mathcal{I}_{J^{\mathfrak{s}}}(\tau) = K \cdot \mathcal{I}_{\widetilde{M}\cap J^{\mathfrak{s}}}(\widetilde{\tau}) \cdot K.$$

On the other side, equation (37) and Lemma 5 imply that

$$\mathcal{I}_{\widetilde{M}\cap J^{\mathfrak{s}}}(\widetilde{\tau}) = \widetilde{K} \subset K.$$

Hence $\mathcal{I}_{J^{\mathfrak{s}}}(\tau) = K$, and the result follows from Proposition 2. $\square$

6.5 The general case

The Levi subgroup $\widetilde{M}$ defined in the beginning of the subsection 6.2 is the G-stabilizer of a decomposition

$$V = \widetilde{V}^1 \oplus \widetilde{V}^2 \oplus + \cdots \oplus \widetilde{V}^t,$$

of V as a direct sum of nonzero subspaces $\widetilde{V}^j$.

Since the endo-class of a supercuspidal representation only depends on the corresponding point in the Bernstein spectrum (see [10, Proposition 4.5]), we can associate to each $\widetilde{V}^j$ an endo-class of simple characters, namely Θ_{π_l} for any l such that $V^l \subset \widetilde{V}^j$.

Now let $\bar{M} \supset M$ be the Levi subgroup in G defined as in [10, §8.1], that is, for each i, let $\bar{V}^i$ be the sum of those $\widetilde{V}^j$ whose associate endo-class Θ_{π_j} is $\Theta(i)$, and write $\bar{M}$ for the G-stabilizer of a decomposition

$$V = \bar{V}^1 \oplus \bar{V}^2 \oplus \cdots \oplus \bar{V}^q.$$

Setting $\bar{N}_i := \dim_F \bar{V}^i$, we get

$$\bar{M} \cong \mathrm{GL}(\bar{N}_1, F) \times \cdots \times \mathrm{GL}(\bar{N}_q, F).$$

We put

$$\bar{K} := K_1 \times K_2 \times \cdots \times K_q \quad \text{and} \quad \bar{\tau} := \tau_1 \times \tau_2 \times \cdots \times \tau_q, \tag{49}$$

where the pairs (K_i, τ_i) are defined as in (45), (47). Then a similar proof as those of [10, §7.2] shows that the pair $(\bar{K}, \bar{\tau})$ is a $\bar{M}$-cover of (J_M, τ_M).

For each $i \in \{1, \ldots, q\}$, let $\bar{\mathcal{L}}^i$, $\bar{\mathcal{L}}^i_{\max}$ respectively denote the $\mathfrak{o}_E$-lattice chain in the E-vector space $\bar{V}^i$ defined by (39), and the $\mathfrak{o}_F$-lattice chain in the F-vector space $\bar{V}^i$ defined by (40). Let m_i denote the number of representations π_l $(1 \leq l \leq m)$ with endo-class θ_i. Then $\bar{\mathcal{L}}^i$, considered as an $\mathfrak{o}_F$-lattice chain, has period $e_i := e(\bar{\mathcal{L}}^i) = e(E_i|F)\, m_i$, and $\bar{\mathcal{L}}^i_{\max}$ has period $e(\bar{\mathcal{L}}^i_{\max}) = m_i$.

Then let Λ^i (resp. $\Lambda^i_{\max}$) denote the (strict) *lattice sequence* defined by the lattice chain $\bar{\mathcal{L}}^i$ (resp. $\bar{\mathcal{L}}^i_{\max}$), considered as $\mathfrak{o}_F$-lattice chains. Then, using the addition of lattice sequences recalled in (15), we define

$$\Lambda := \Lambda^1 \oplus \Lambda^2 \oplus \cdots \oplus \Lambda^q, \tag{50}$$

and

$$\Lambda_{\max} := e(E_1|F)\Lambda^1_{\max} \oplus e(E_2|F)\Lambda^2_{\max} \oplus \cdots \oplus e(E_q|F)\Lambda^q_{\max}. \tag{51}$$

Let $\mathcal{L}_\Lambda$, $\mathcal{L}_{\Lambda_{\max}}$ denote the $\mathfrak{o}_F$-lattice chains attached to the lattice sequences Λ, $\Lambda_{\max}$, respectively, as in (11). Let $\mathfrak{A}_\Lambda$, $\mathfrak{A}_{\Lambda_{\max}}$ denote the hereditary $\mathfrak{o}_F$-orders in A defined by the lattice chain $\mathcal{L}_\Lambda$, $\mathcal{L}_{\Lambda_{\max}}$, respectively. We have (see [10, Proposition 2.3. (i)]):

$$\mathfrak{A}_\Lambda = \mathfrak{A}(\mathcal{L}_\Lambda) = \mathfrak{a}_0(\Lambda) \quad \text{and} \quad \mathfrak{A}_{\Lambda_{\max}} = \mathfrak{A}(\mathcal{L}_{\Lambda_{\max}}) = \mathfrak{a}_0(\Lambda_{\max}), \tag{52}$$

where $\mathfrak{a}_0(\Lambda)$, $\mathfrak{a}_0(\Lambda_{\max})$ are defined as in (8).

Lemma 8. *The $\mathfrak{o}_F$-order $\mathfrak{A}_\Lambda$ is contained in the $\mathfrak{o}_F$-order $\mathfrak{A}_{\Lambda_{\max}}$.*

Proof. Let $e := \operatorname{lcm}\{e_1, \dots, e_q\}$. Both Λ and $\Lambda_{\max}$ have period e. ¿From (15), we have

$$\Lambda(ex) = \Lambda^1(e_1x) \oplus \cdots \oplus \Lambda^q(e_qx), \quad \Lambda_{\max}(ex) = \Lambda^1_{\max}(e_1x) \oplus \cdots \oplus \Lambda^q_{\max}(e_qx),$$

for each $x \in \mathbb{R}$. On the other side, (12) gives

$$\Lambda^i\left(\frac{e_i}{e}j\right) = \Lambda(l_i(j)) \quad \text{and} \quad \Lambda^i_{\max}\left(\frac{e_i}{e}j\right) = \Lambda_{\max}(l_i(j)),$$

for each $j \in \mathbb{Z}$, where $l_i(j)$ is the integer defined by the relation

$$l_i(j) - 1 < \frac{e_i}{e}j \leq l_i(j).$$

Hence

$$\Lambda(j) = \Lambda^1(l_1(j)) \oplus \cdots \oplus \Lambda^q(l_q(j)), \quad \Lambda_{\max}(j) = \Lambda^1_{\max}(l_1(j)) \oplus \cdots \oplus \Lambda^q_{\max}(l_q(j)).$$

Then the result is consequence of Lemma 6. $\square$

The following definition generalizes Definitions 4 and 5.

Definition 6. We set

$$\mathfrak{A}^{\mathfrak{s}} := \mathfrak{A}_{\Lambda_{\max}}, \quad \text{and} \quad J^{\mathfrak{s}} := U(\mathfrak{A}^{\mathfrak{s}}).$$

Example 2. We assume here that $q = m$, that is, the representations π_l have all distinct endo-classes. It implies that $\bar{M} = \widetilde{M} = M$. Then each lattice sequence $\Lambda^l_{\max}$ has period 1, and so $\Lambda_{\max}$ has also period 1. We get in this case $J^{\mathfrak{s}} = \mathrm{GL}(N, \mathfrak{o}_F)$.

Theorem 6. *There exists a G-cover (J, τ) of (J_M, λ_M) such that*

- $J \subset J^{\mathfrak{s}}$,

- $\alpha := \mathrm{Ind}_J^{J^{\mathfrak{s}}}(\tau)$ *is irreducible. Hence* $(J^{\mathfrak{s}}, \alpha)$ *is an* $\mathfrak{s}$*-type.*

Proof. Let (J, τ) be the G-cover of $(\bar{K}, \bar{\tau})$ constructed in the similar way as in [10, §8], in particular, we have

$$J \subset U(\mathfrak{A}_\Lambda).$$

Then the first assertion follows from Lemma 8.

On the other side the same proof as those of [10, §8.2, Main Theorem] gives the following formula for the intertwining:

$$\mathcal{I}_G(\tau) = J \cdot \mathcal{I}_{\widetilde{M}}(\tau_{\widetilde{M}}) \cdot J.$$

Since $J \subset J^{\mathfrak{s}}$, it implies:

$$\mathcal{I}_{J^{\mathfrak{s}}}(\tau) = J \cdot \mathcal{I}_{\widetilde{M} \cap J^{\mathfrak{s}}}(\tau_{\widetilde{M}}) \cdot J = J \cdot \mathcal{I}_{\widetilde{J}^{\mathfrak{s}}}(\widetilde{\tau}) \cdot J.$$

Now, by Lemma 5, we have

$$\mathcal{I}_{\widetilde{J}^{\mathfrak{s}}}(\tau_{\widetilde{M}}) = \widetilde{K}.$$

We get

$$\mathcal{I}_{J^{\mathfrak{s}}}(\tau) = J,$$

and the result follows from Proposition 2. $\square$

7 Supercuspidal Bernstein components

Let $\mathfrak{s} = [G, \pi]_G$, where $G = \mathrm{GL}(N, F)$. Here π is an irreducible supercuspidal representation of G.

Let (J, λ) be a maximal simple type contained in π, as in [8]. We have $e = 1$ and hence $\mathfrak{A}_{\mathfrak{s}} = \mathrm{M}(N, \mathfrak{o}_F)$. It follows that $J^{\mathfrak{s}} = \mathrm{GL}(N, \mathfrak{o}_F) = L_0$. By Proposition 4, the representation $\alpha = \mathrm{Ind}_J^{L_0}(\lambda)$ is irreducible. The pair (L_0, α) is an $\mathfrak{s}$-type. The restriction to L_0 of a smooth irreducible representation π' of G contains α if and only if π' is isomorphic to $\pi \otimes \chi \circ \det$, where χ is an unramified quasicharacter of $F^\times$. Moreover, π contains α with multiplicity 1. In fact, the representation α is the *unique* smooth irreducible representation τ of L_0 such that (L_0, τ) is an $\mathfrak{s}$-type, see [16].

The little complex $C_*(\mathfrak{s})$ determined by α is

$$0 \longleftarrow C_0(\mathfrak{s}) \longleftarrow 0$$

where $C_0(\mathfrak{s})$ is the free abelian group on the invariant 0-cycle

$$(\tau, \mathcal{R}(\tau), \mathcal{R}^2(\tau), \ldots, \mathcal{R}^{n-1}(\tau))$$

The total homology of the little complex is given by $h_0(\mathfrak{s}) = \mathbb{Z}$. Therefore, by Lemma 1, we have

$$H_{\mathrm{ev}}(\mathfrak{s}) = \mathbb{Z} = H_{\mathrm{odd}}(\mathfrak{s}).$$

Theorem 7. *Let π be an irreducible unitary supercuspidal representation of* $\mathrm{GL}(N)$. *Let $\mathfrak{s} = [G, \pi]_G$. Then we have*

$$H_{\mathrm{ev}}(\mathfrak{s}) \cong K_0(\mathcal{A}^{\mathfrak{s}}), \quad H_{\mathrm{odd}}(\mathfrak{s}) \cong K_1(\mathcal{A}^{\mathfrak{s}}).$$

Proof. The C^*-ideal $\mathcal{A}^{\mathfrak{s}}$ is given by

$$\mathcal{A}^{\mathfrak{s}} = C(S^1, \mathfrak{K})$$

where $\mathfrak{K}$ is the C^*-algebra of compact operators and

$$S^1 = \{\pi \otimes \chi \circ \det \; : \chi \in (F^\times)^\wedge\}.$$

The noncommutative C^*-algebra $\mathcal{A}^{\mathfrak{s}}$ is strongly Morita equivalent to the commutative C^*-algebra $C(S^1)$. For this C^*-algebra we have

$$K_j(C(S^1)) \cong K^j(S^1) = \mathbb{Z}$$

where $j = 0, 1$.

$\square$

8 Generic Bernstein components attached to a maximal Levi subgroup

We assume in this section that $\mathfrak{s} = [M, \sigma]_G$ with $M \cong \mathrm{GL}(N_1) \times \mathrm{GL}(N_2)$ a 2-blocks Levi subgroup of G such that $W_{\mathfrak{t}} = \{1\}$. Note that the last conditions is always satisfied if $N_1 \neq N_2$.

Let (J_M, λ_M) be an $\mathfrak{t}$-type and let (J, τ) be the G-cover of (J_M, τ_M) considered in Theorem 6. We have shown there that $\alpha := \mathrm{Ind}_J^{J^{\mathfrak{s}}}(\tau)$ is irreducible. It then follows from Propositions 2 and 5 that $\beta = \mathrm{Ind}_J^{L_0}(\tau)$ is irreducible.

Let $C_0(\tau)$, $C_1(\tau)$ denote respectively the free abelian group on one generator $(\beta, \mathcal{R}(\beta), \ldots, \mathcal{R}^{N-1}(\beta)$, and on $(\alpha, \mathcal{R}(\alpha), \ldots, \mathcal{R}^{N-1}\alpha)$. The little complex is

$$0 \longleftarrow C_0(\mathfrak{s}) \xleftarrow{\partial} C_1(\mathfrak{s}) \longleftarrow 0$$

The map ∂ is 0 by vertex compatibility of $(\alpha, \mathcal{R}(\alpha), \ldots, \mathcal{R}^{N-1}(\alpha))$. Then $h_0(\mathfrak{s}) = \mathbb{Z}, h_1(\mathfrak{s}) = \mathbb{Z}$ and so $H_{\mathrm{ev}}(\mathfrak{s}) = \mathbb{Z}^2 = H_{\mathrm{odd}}(\mathfrak{s})$.

The subset of the tempered dual of $\mathrm{GL}(N)$ which contains the $\mathfrak{s}$-type (J, τ) has the structure of a compact 2-torus. But $K^0(\mathbb{T}^2) = \mathbb{Z}^2 = K^1(\mathbb{T}^2)$ as required.

Theorem 8. *The $\mathfrak{s}$-type (J, τ) generates a little complex $C(\mathfrak{s})$. For this complex we have*

$$H_{\mathrm{ev}}(\mathfrak{s}) \cong K_0(\mathcal{A}^{\mathfrak{s}}) = \mathbb{Z}^2, \quad H_{\mathrm{odd}}(\mathfrak{s}) \cong K_1(\mathcal{A}^{\mathfrak{s}}) = \mathbb{Z}^2$$

Note that the above Theorem applies to the intermediate principal series of GL(3). In the next section, we will consider the principal series of GL(3).

9 Principal series in GL(3)

Here s_0, s_1, s_2 are the standard involutions

$$s_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad s_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad s_0 = \begin{pmatrix} 0 & 0 & \varpi^{-1} \\ 0 & 1 & 0 \\ \varpi & 0 & 0 \end{pmatrix}$$

where

$$\Pi = \Pi_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varpi & 0 & 0 \end{pmatrix}.$$

Note that $\mathrm{val}(\det(\Pi)) = 1$. Restricted to the affine line $\mathbb{R}$ in the enlarged building $\beta^1\mathrm{GL}(3) = \beta\mathrm{SL}(3) \times \mathbb{R}$, Π sends t to $t+1$. We also have $\Pi^3 = \varpi 1 \in \mathrm{GL}(3)$.

We have the double coset identities

$$0 \le k \le 2 \Longrightarrow I\backslash J_k/I = \{1, s_k\} \tag{53}$$

$$r, s, t \text{ distinct } \Longrightarrow J_r\backslash L_s/J_r = \{1, s_t\}. \tag{54}$$

Let $\mathfrak{s} = [T, \sigma]_G$, where T is the diagonal split torus in GL(3):

$$T = \begin{pmatrix} F^\times & 0 & 0 \\ 0 & F^\times & 0 \\ 0 & 0 & F^\times \end{pmatrix},$$

and σ is an irreducible smooth character of T.

9.1. Construction of an $\mathfrak{s}$-type, following Roche

For $u \in F$, we set

$$x_{1,2}(u) = \begin{pmatrix} 1 & u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_{1,3}(u) = \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_{2,3}(u) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix},$$

$$x_{2,1}(u) = \begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_{3,1}(u) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u & 0 & 1 \end{pmatrix}, \quad x_{3,2}(u) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & u & 1 \end{pmatrix},$$

and, for any $k \in \mathbb{Z}$,

$$U_{i,j,k} = x_{i,j}(\mathfrak{p}_F^k).$$

Let $\Phi = \{\alpha_{i,j} : 1 \leq i, j \leq 2\}$ be the set of roots of G with respect to T. For each root $\alpha_{i,j}$, let $\alpha_{i,j}^\vee$ denotes the corresponding coroot. We have

$$\alpha_{1,2}^\vee(t) = \begin{pmatrix} t^{-1} & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \alpha_{2,1}^\vee(t) = \begin{pmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\alpha_{1,3}^\vee(t) = \begin{pmatrix} t^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{pmatrix}, \quad \alpha_{3,1}^\vee(t) = \begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{-1} \end{pmatrix},$$

$$\alpha_{2,3}^\vee(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & t \end{pmatrix}, \quad \alpha_{3,2}^\vee(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-1} \end{pmatrix}.$$

Define $\sigma \colon T \to \mathbb{T}$ by

$$\sigma \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = \sigma_1(a)\sigma_2(b)\sigma_3(c),$$

where $\sigma_i \colon F^\times \to \mathbb{T}$ is a character of $F^\times$, for $i = 1, 2, 3$.

Hence $\sigma \circ \alpha_{i,j}^\vee \colon \mathfrak{o}_F^\times \to \mathbb{T}$ is the smooth character of $\mathfrak{o}_F^\times$ defined by

$$\sigma \circ \alpha_{i,j}^\vee(t) = \sigma_j(t)\sigma_i(t^{-1}) = (\sigma_j \sigma_i^{-1})(t).$$

Now if $\chi \colon \mathfrak{o}_F^\times \to \mathbb{T}$ is a smooth character, let $c(\chi)$ be the conductor of χ: the least integer $n \geq 1$ such that $1 + \mathfrak{p}_F^n \subset \ker(\chi)$. We will write $c_{i,j}$ for $c(\sigma \circ \alpha_{i,j}^\vee)$. We get

$$c_{i,j} = c(\sigma_j/\sigma_i) = c_{j,i}.$$

We can define a function $f = f_\sigma \colon \Phi \to \mathbb{Z}$ (here Φ is the set of roots) as follows:

$$f_\sigma(\alpha_{i,j}) = \begin{cases} [c_{i,j}/2] & \text{if } \alpha_{i,j} \in \Phi^+, \\ [(c_{i,j}+1)/2] & \text{if } \alpha_{i,j} \in \Phi^-. \end{cases}$$

Here $[x]$ denotes the largest integer $\leq x$.

Let

$$U_\sigma = \langle U_{i,j,f(\alpha_{i,j})} \,:\, \alpha_{i,j} \in \Phi \rangle,$$

and

$$J = \langle {}^\circ T, U_\sigma \rangle = {}^\circ T U_\sigma = U_\sigma {}^\circ T,$$

where ${}^\circ T$ is the compact part of T,

$${}^\circ T = \begin{pmatrix} \mathfrak{o}_F^\times & 0 & 0 \\ 0 & \mathfrak{o}_F^\times & 0 \\ 0 & 0 & \mathfrak{o}_F^\times \end{pmatrix}.$$

It follows that

$$J = \begin{pmatrix} \mathfrak{o}_F^\times & \mathfrak{p}_F^{[c_{1,2}/2]} & \mathfrak{p}_F^{[c_{1,3}/2]} \\ \mathfrak{p}_F^{[(c_{1,2}+1)/2]} & \mathfrak{o}_F^\times & \mathfrak{p}_F^{[c_{2,3}/2]} \\ \mathfrak{p}_F^{[(c_{1,3}+1)/2]} & \mathfrak{p}_F^{[(c_{2,3}+1)/2]} & \mathfrak{o}_F^\times \end{pmatrix}.$$

The group J will give the open compact group we are looking for.

Next, we need to figure out what is the correct character of J. In order to do that, we set

$$T_\sigma = \prod_{\alpha_{i,j} \in \Phi} \alpha_{i,j}^\vee (1 + \mathfrak{p}_F^{f(\alpha_{i,j}) + f(-\alpha_{i,j})}) \subset {}^\circ T.$$

Setting

$$U_\sigma^+ = U_\sigma \cap \begin{pmatrix} 1 & F & F \\ 0 & 1 & F \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad U_\sigma^- = U_\sigma \cap \begin{pmatrix} 1 & 0 & 0 \\ F & 1 & 0 \\ F & F & 1 \end{pmatrix},$$

we obtain

$$U_\sigma = U_\sigma^- \cdot T_\sigma \cdot U_\sigma^+ \quad \text{and} \quad J = U_\sigma^- \cdot {}^\circ T \cdot U_\sigma^+.$$

It follows that

$$J/U_\sigma \cong {}^\circ T / T_\sigma.$$

By construction, $T_\sigma \subset \ker(\sigma_{|{}^\circ T})$. Hence $\sigma_{|{}^\circ T}$ defines a character of ${}^\circ T/T_\sigma$, and so can be lifted to a character τ of J. Then (J, τ) an $\mathfrak{s}$-type by [18, Theorem 7.7].

9.2 Intertwining

We first recall that the following results ([18, Theorem 4.15])

$$I_G(\tau) = J \,\widetilde{W}(\sigma)\, J, \tag{55}$$

where
$$\widetilde{W}(\sigma) = \left\{ v \in \widetilde{W} \;:\; {}^v\sigma = \sigma \right\}.$$
More generally, it follows by the same proof as those of [18, Theorem 4.15], using [1, Prop. 9.3] instead of [18, Prop. 4.11], that, for each $w \in W$,
$$I_G(\tau, {}^\tau) = J\,\widetilde{W}(\sigma, {}^w\sigma)\,{}^wJ, \tag{56}$$
where
$$\widetilde{W}(\sigma, {}^w\sigma) = \left\{ v \in \widetilde{W} \;:\; {}^v\sigma = {}^w\sigma \right\}.$$
Let
$$\Phi(\sigma) = \{\alpha_{i,j} \in \Phi \;:\; (\sigma_i)_{|}\mathfrak{o}_F^\times = (\sigma_j)_{|}\mathfrak{o}_F^\times\} \subset \Phi.$$
The group $W_0(\sigma)$ is equal to the group $W_{\mathfrak{s}_T}$, where $\mathfrak{s}_T = [T, \lambda]_T$. We observe that
$$I_{L_0}(\tau) = J\,W_0(\sigma)\,J. \tag{57}$$

9.2.1 The case $\Phi(\sigma) = \Phi$. Let $\mathfrak{s} = [T, \sigma]_G$, where $\sigma = \psi \circ \det$ with ψ a smooth character of $F^\times$. In this case $c_{i,j} = 1$ for any i, j. It follows that $J = I$.

The pair (I, τ) is an $\mathfrak{s}$-type. We will construct cycles from this type. It follows from (57) that, as $\mathbb{C}$-algebras,
$$\mathrm{End}_{L_0}\,(\mathrm{Ind}_I^{L_0}\tau) \cong \mathcal{H}(\mathrm{GL}(3, k_F)//B).$$
We also have, as $\mathbb{C}$-algebras,
$$\mathcal{H}(\mathrm{GL}(3, k_F)//B) \cong \mathbb{C}[W_0] \cong \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$$
so that
$$\mathrm{Ind}_I^{L_0}\tau = \lambda_{L_0} \oplus \mu_{L_0} \oplus \nu_{L_0} \oplus \nu_{L_0}$$
where $\lambda_{L_0}, \mu_{L_0}, \nu_{L_0}$ are distinct.

We also have
$$\sigma|J_0 \hookrightarrow \mathrm{Ind}_I^{J_0}\tau$$
by Frobenius reciprocity. The triple $(\sigma|J_0, \mathcal{R}(\sigma|J_0), \mathcal{R}^2(\sigma|J_0))$ is an invariant 1-cycle, and is not the boundary of 1_I.

We now form the little complex:

- $C_0(\mathfrak{s})$ is the free abelian group on the three invariant 0-cycles
$$\lambda_L := (\lambda_{L_0}, \mathcal{R}(\lambda_{L_0}), \mathcal{R}^2(\lambda_{L_0}))$$
$$\mu_L := (\mu_{L_0}, \mathcal{R}(\mu_{L_0}), \mathcal{R}^2(\mu_{L_0}))$$
$$\nu_L := (\nu_{L_0}, \mathcal{R}(\nu_{L_0}), \mathcal{R}^2(\nu_{L_0}))$$

- $C_1(\mathfrak{s})$ is the free abelian group on the invariant 1-cycle

$$\lambda_J := (\sigma|J_0, \mathcal{R}(\sigma|J_0), \mathcal{R}^2(\sigma|J_0))$$

In the little complex

$$0 \longleftarrow C_0(\mathfrak{s}) \xleftarrow{0} C_1(\mathfrak{s}) \longleftarrow 0$$

we have

$$h_0(\mathfrak{s}) = \mathbb{Z}^3, \ h_1(\mathfrak{s}) = \mathbb{Z}.$$

The total homology of the little complex is $\mathbb{Z}^4$. As generating cycles we may take

$$\lambda_L, \mu_L, \nu_L, \lambda_J.$$

and so, by Lemma 1, the even (resp. odd) chamber homology groups are

$$H_{\text{ev}}(\mathfrak{s}) = \mathbb{Z}^4, \quad H_{\text{odd}}(\mathfrak{s}) = \mathbb{Z}^4.$$

Each irreducible representation ρ of a compact open subgroup J creates an *idempotent* in $\mathcal{A}$ as follows. Let d denote the dimension of ρ, let χ denote the character of ρ. Form the function $d \cdot \chi : J \longrightarrow \mathbb{C}$ and *extend by* 0 to G. This function on G is a non-zero idempotent in $\mathcal{A}$, with the convolution product. We will denote this idempotent by $e(\rho)$:

$$e(\rho) * e(\rho) = e(\rho).$$

The inclusion

$$H_{\text{ev}}(\mathfrak{s}) \hookrightarrow K_0(\mathcal{A})$$

is given explicitly as follows:

$$\lambda_L \mapsto e(\lambda_{L_1}), \mu_L \mapsto e(\mu_{L_1}), \nu_L \mapsto e(\nu_{L_1}), \lambda_J \mapsto e(\lambda_{J_1}).$$

It follows from [17] that the C^*-ideal $\mathcal{A}^{\mathfrak{s}}$ is given as follows:

$$\mathcal{A}^{\mathfrak{s}} \cong C(\text{ Sym}^3\, \mathbb{T}, \mathfrak{K}) \oplus C(\mathbb{T}^2, \mathfrak{K}) \oplus C(\mathbb{T}, \mathfrak{K}).$$

The symmetric cube $\text{Sym}^3\, \mathbb{T}$ is homotopy equivalent to $\mathbb{T}$ via the product map

$$\text{Sym}^3\, \mathbb{T} \sim \mathbb{T}, \quad (z_1, z_2, z_3) \mapsto z_1 z_2 z_3.$$

Hence $K_0(\mathcal{A}^{\mathfrak{s}}) = \mathbb{Z}^4 = K_1(\mathcal{A}^{\mathfrak{s}})$ as required.

Note that

- $\text{Sym}^3\mathbb{T}$ is in the minimal unitary principal series of GL(3)

- $\mathbb{T}^2$ is in the intermediate unitary principal series of GL(3)
- $\mathbb{T}$ is in the discrete series of GL(3); if $\tau = 1$ then $\mathbb{T}$ comprises the unramified unitary twists of the Steinberg representation of GL(3)

These are precisely the tempered representations of GL(3) which contain the type (I, τ).

Theorem 9. *Let $\mathfrak{s} = [T, \sigma]_G$ where $\sigma = \psi \circ \det$ and ψ is a smooth (unitary) character of $F^\times$. Then we have*

$$H_{\text{ev}}(\mathfrak{s}) \cong K_0(\mathcal{A}^\mathfrak{s}) = \mathbb{Z}^4, \quad H_{\text{odd}}(\mathfrak{s}) \cong K_1(\mathcal{A}^\mathfrak{s}) = \mathbb{Z}^4.$$

9.2.2 The case $\emptyset \neq \Phi(\sigma) \neq \Phi$

Assume that $(\sigma_1)_{|\mathfrak{o}_F^\times} = (\sigma_2)_{|\mathfrak{o}_F^\times} \neq (\sigma_3)_{|\mathfrak{o}_F^\times}$. We have

$$J = \begin{pmatrix} \mathfrak{o}_F^\times & \mathfrak{o}_F & \mathfrak{p}_F^{[\ell/2]} \\ \mathfrak{p}_F & \mathfrak{o}_F^\times & \mathfrak{p}_F^{[\ell/2]} \\ \mathfrak{p}_F^{[(\ell+1)/2]} & \mathfrak{p}_F^{[(\ell+1)/2]} & \mathfrak{o}_F^\times \end{pmatrix},$$

where $\ell = c_{1,3} = c_{2,3}$, and

$$\tau \begin{pmatrix} a & * & * \\ * & b & * \\ * & * & c \end{pmatrix} = \sigma_1(a)\sigma_1(b)\sigma_3(c).$$

It is clear that $s_1 \in I_{L_0}(\tau)$. The Weyl group $W_{\mathfrak{s}_T} = \mathbb{Z}/2\mathbb{Z}$ and so we have $I_{L_o}(\tau) = J \cup J s_1 J$. The complete list is as follows:

$$I_I(\tau) = J$$

$$I_{J_1}(\tau) = J < s_1 > J, \quad I_{J_2}(\tau) = J, \quad I_{J_0}(\tau) = J$$

$$I_{L_1}(\tau) = J < s' > J, \quad I_{L_2}(\tau) = J < s_1 > J, \quad I_{L_0}(\tau) = J < s_1 > J$$

where

$$s' = \begin{pmatrix} 0 & \varpi^{-1} & 0 \\ \varpi & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Lemma 9. *Let $\tau_1 = \mathrm{Ind}_J^I(\tau)$. Then τ_1 is irreducible.*

Proof. This follows from proposition 2, since $I_I(\tau) = J$. It follows that (I, τ_1) is an $\mathfrak{s}$-type. $\square$

Lemma 10. *We have*

$$\mathrm{Ind}_I^{J_1}\tau_1 = \xi_1 \oplus \eta_1, \quad \mathrm{Ind}_I^{L_0}\tau_1 = \gamma_0 \oplus \delta_0.$$

Proof. We have $I_{J_1}(\tau) = J \cup s_1 J$. Hence

$$\mathrm{End}_{J_1}(\mathrm{Ind}_I^{J_1}\tau_1) = \mathcal{I}_1(\tau) \oplus \mathcal{I}_{s_1}(\tau) = \mathbb{C} \oplus \mathbb{C}.$$

This implies that $\mathrm{Ind}_I^{J_1}\tau_1$ has two distinct irreducible constituents $\xi_1,\ \eta_1$. Now, we replace J_1 by L_0, and infer that $\mathrm{Ind}_I^{L_0}\tau_1$ has two distinct irreducible constituents γ_0, δ_0. $\square$

It follows that

$$\mathrm{Ind}_I^{J_2}\mathcal{R}(\tau_1) = \mathcal{R}(\xi_1) \oplus \mathcal{R}(\eta_1),$$
$$\mathrm{Ind}_I^{J_0}\mathcal{R}^2(\tau_1) = \mathcal{R}^2(\xi_1) \oplus \mathcal{R}^2(\eta_1).$$

This creates two invariant 1-chains

$$\xi := (\xi_1, \mathcal{R}(\xi_1), \mathcal{R}^2(\xi_1)),\ \eta := (\eta_1, \mathcal{R}(\eta_1), \mathcal{R}^2(\eta_1)).$$

It follows from (5) that

$$\mathrm{Ind}_I^{L_0}\tau_1 \cong \mathrm{Ind}_I^{L_0}\mathcal{R}(\tau_1)$$

$$\zeta_1 := \mathrm{Ind}_I^{J_1}\mathcal{R}(\tau_1) \cong \mathrm{Ind}_I^{J_1}\mathcal{R}^2(\tau_1)$$

By (17) we have

$$0 = \langle \mathrm{Ind}_J^{J_1}\tau, \mathrm{Ind}_J^{J_1}\mathcal{R}(\tau)\rangle.$$

Let $C_0(\mathfrak{s})$ be the free abelian group generated by the two invariant 0-cycles

$$(\gamma_0, \mathcal{R}(\gamma_0), \mathcal{R}^2(\gamma_0)),\ (\delta_0, \mathcal{R}(\delta_0), \mathcal{R}^2(\delta_0)).$$

Let $C_1(\mathfrak{s})$ be the free abelian group generated by the two invariant 1-cycles ξ and ζ.

The little complex is then

$$0 \longleftarrow C_0(\mathfrak{s}) \xleftarrow{0} C_1(\mathfrak{s}) \longleftarrow 0.$$

We have $h_0(\mathfrak{s}) = \mathbb{Z}^2, h_1(\mathfrak{s}) = \mathbb{Z}^2$ and the total homology is $\mathbb{Z}^4$ and so $H_{\mathrm{ev}}(\mathfrak{s}) = \mathbb{Z}^4 = H_{\mathrm{odd}}(\mathfrak{s})$.

The definition of $\zeta := (\zeta_0, \mathcal{R}(\zeta_0), \mathcal{R}^2(\zeta_0))$ shows that

$$\partial(\tau_1 + \mathcal{R}(\tau_1) + \mathcal{R}^2(\tau_1)) = \xi + \eta + 2\zeta$$

so that η and $-(\xi + 2\zeta)$ are homologous. Therefore the invariant 1-cycle η does not contribute a new homology class in $H_1(G; \beta^1 G)$.

The C^*-ideal $\mathcal{A}^{\mathfrak{s}}$ is as follows:

$$C(\mathbb{T}^2, \mathfrak{K}) \oplus C(\mathrm{Sym}^2\mathbb{T} \times \mathbb{T}, \mathfrak{K}).$$

To identify these ideals, we proceed as follows. First, let $\Psi(F^\times)$ denote the group of unramified unitary characters of $F^\times$. The first summand is determined by the compact orbit

$$\mathcal{O}(\mathrm{St}(\sigma_1, 2) \otimes \sigma_3) = \{\chi_1 \mathrm{St}(\sigma_1, 2) \otimes \chi_2\sigma_3 : \chi_j \in \Psi(F^\times)\}$$

where $\mathrm{St}(\sigma_1, 2)$ is a generalized Steinberg representation; the second is determined by the compact orbit

$$\mathcal{O}(\sigma_1 \otimes \sigma_1 \otimes \sigma_3) = \{\chi_1\sigma_1 \otimes \chi_2\sigma_1 \otimes \chi_3\sigma_3 : \chi_j \in \Psi(F^\times)\}.$$

The compact space $\mathrm{Sym}^2\mathbb{T} \times \mathbb{T}$ is homotopy equivalent to the 2-torus $\mathbb{T}^2$.

The space $\mathrm{Sym}^2\mathbb{T} \times \mathbb{T}$ is in the minimal unitary principal series of GL(3) and the space $\mathbb{T}^2$ is in the intermediate unitary principal series of GL(3). The union of these two compact spaces is precisely the set of tempered representations of GL(3) which contain the $\mathfrak{s}$-type (J, τ).

The K-groups are now immediate:

$$K_j(\mathcal{A}^{\mathfrak{s}}) = \mathbb{Z}^4$$

with $j = 0, 1$.

Theorem 10. *Let $\mathfrak{s} = [T, \sigma]_G$. We have*

$$H_{\mathrm{ev}}(\mathfrak{s}) \cong K_0(\mathcal{A}^{\mathfrak{s}}) = \mathbb{Z}^4, \quad H_{\mathrm{odd}}(\mathfrak{s}) \cong K_1(\mathcal{A}^{\mathfrak{s}}) = \mathbb{Z}^4.$$

9.2.3 The case $\Phi(\sigma) = \emptyset$. The generic torus. The Bernstein component is $[T, \sigma_1 \otimes \sigma_2 \otimes \sigma_3]$. The Weyl group $W(T) = W_0 = S_3$, and the associated parahoric subgroup is the Iwahori subgroup I.

The restrictions of σ_1, σ_2 and σ_3 to $\mathfrak{o}_F^\times$ are all distinct. We have $\Phi(\sigma) = \emptyset$. We have $\widetilde{W}(\sigma) = D$, where D is the subgroup of T whose eigenvalues are powers of ϖ. The subgroup D is free abelian of rank 3. The only compact element in D is 1_G. The only double-J-coset representative in L_0 which G-intertwines τ is 1_G. This proves the following:

Lemma 11. *If $r = 0, 1, 2$ then $\mathrm{Ind}_J^{J_r}(\tau)$ is irreducible, $\mathrm{Ind}_J^{L_r}(\tau)$ is irreducible.*

Let $\alpha = \mathrm{Ind}_J^I(\tau)$. Then α is irreducible. Therefore (I, α) is an $\mathfrak{s}$-type.

Lemma 12. *If $w \in W_0$ then* $\mathrm{Ind}_I^{L_0}\alpha = \mathrm{Ind}_I^{L_0}({}^w\alpha)$.

Proof. We have $\mathrm{Ind}_I^{L_0}(\alpha) = \mathrm{Ind}_J^{L_0}(\tau)$ and $\mathrm{Ind}_I^{L_0}({}^w\alpha) = \mathrm{Ind}_J^{L_0}({}^w\tau)$. By Proposition 3, it is sufficient to prove that $I_G(\tau, {}^w\tau) \neq \{0\}$. But $I_G(\tau, {}^w\tau) = J\,\widetilde{W}(\sigma, {}^w\sigma)\,J$. □

Lemma 13. *If $w \in W_0$ then*

$$\mathrm{Ind}_I^{J_r}(\alpha) \cong \mathrm{Ind}_I^{J_r}({}^w\alpha) \Longleftrightarrow w \in < s_r >$$

with $0 \leq r \leq 2$.

Proof. By Proposition 3,

$$\mathrm{Ind}_J^{J_r}(\tau) \cong \mathrm{Ind}_J^{J_r}({}^w\tau) \Longleftrightarrow I_{J_r}(\tau, {}^w\tau) \neq \{0\}.$$

From (56), we have

$$I_{J_r}(\tau, {}^w\tau) = J_r \cap \widetilde{W}(\sigma, {}^w\sigma) = J_r \cap \widetilde{W}(\sigma) \cdot w = J_r \cap D \cdot w.$$

The result follows from the fact that $J_r = I\ < 1, s_r >\ I$. □

Inducing the orbit $W_0 \cdot \alpha$ from J to J_1 gives 3 distinct elements ρ_1, ϕ_1, ψ_1, by Lemma 9. Inducing from J to L_0 gives γ_0.

Set $C_2(\mathfrak{s})$ = free abelian group on the invariant 2-cycle

$$\epsilon := \sum_{w \in W_0} sgn(w)({}^w\alpha).$$

Set $C_1(\mathfrak{s})$ = free abelian group on the three invariant 1-cycles

$$\rho := (\rho_1, \mathcal{R}(\rho_1), \mathcal{R}^2(\rho_1)),$$

$$\phi := (\phi_1, \mathcal{R}(\phi_1), \mathcal{R}^2(\phi_1)),$$

$$\psi := (\psi_1, \mathcal{R}(\psi_1), \mathcal{R}^2(\psi_1)).$$

Set $C_0(\mathfrak{s})$ = free vector abelian group on the invariant 0-cycle

$$\gamma := (\gamma_0, \mathcal{R}(\gamma_0), \mathcal{R}^2(\gamma_0)).$$

Note that

$$\partial(\sum_{w \in Alt(3)} {}^w\alpha) = \rho + \phi + \psi$$

where $Alt(3)$ is the alternating subgroup of W_0. Since ${}^{s_1 s_2}\alpha = \mathcal{R}(\alpha)$, we may also write this as

$$\partial(\alpha + \mathcal{R}(\alpha) + \mathcal{R}^2(\alpha)) = \rho + \phi + \psi.$$

It follows that ψ is homologous to $-(\rho+\phi)$ in the top row of the double complex C_{**}. This implies that the *image* of $C(\mathfrak{s})$ in C_{**} determines 4 homology classes. As representing cycles we may take the 2-cycle ϵ, the two 1-cycles ρ, ϕ, and the 0-cycle γ. Therefore

$$\mathrm{H}_{\mathrm{ev}}(G; \beta^1 G)^{\mathfrak{s}} = \mathrm{H}_{\mathrm{odd}}(G; \beta^1 G)^{\mathfrak{s}} = \mathbb{Z}^4.$$

Theorem 11. *The subspace of the tempered dual of* GL(3) *which contains the* $\mathfrak{s}$*-type* (I, α) *has the structure of a compact 3-torus. This is a generic torus in the minimal unitary principal series of* GL(3). *We have*

$$\mathrm{H}_{\mathrm{ev}}(G; \beta^1 G)^{\mathfrak{s}} \cong K_0(\mathcal{A}^{\mathfrak{s}}) = \mathbb{Z}^4, \quad \mathrm{H}_{\mathrm{odd}}(G; \beta^1 G)^{\mathfrak{s}} \cong K_1(\mathcal{A}^{\mathfrak{s}}) = \mathbb{Z}^4.$$

Proof. Let $\Psi(F^\times)$ denote the group of unramified characters of $F^\times$. If $\chi \in \Psi(F^\times)$ then $\chi(x) = z^{val(x)}$ with z a complex number of modulus 1, so that

$$\Psi(F^\times) \cong \mathbb{T}.$$

Writing

$$\mathbb{T}^3 = \{\mathrm{Ind}_T^G(\chi_1\sigma_1 \otimes \chi_2\sigma_2 \otimes \chi_3\sigma_3) : \chi_j \in \Psi(F^\times)\}$$

we have

$$\mathcal{A}^{\mathfrak{s}} \cong C(\mathbb{T}^3, \mathfrak{K})$$

which is strongly Morita equivalent to $C(\mathbb{T}^3)$. The K-theory of the 3-torus is given by

$$K^j(\mathbb{T}^3) = \mathbb{Z}^4$$

where $j = 0, 1$. □

A Chamber homology and K-theory

Let $G = \mathrm{GL}(N)$ and let $\mathcal{A}$ denote the reduced C^*-algebra of G. Let $\mathcal{H}(G)$ be the convolution algebra of uniformly locally constant, compactly supported, complex-valued functions on G, and let $\mathcal{C}(G)$ be the Harish-Chandra Schwartz algebra of G. The following diagram serves as a framework for this

article:

$$\begin{tikzcd}
K_j^{\mathrm{top}}(G) \arrow[rr, "\mu"] \arrow[d, "\mathrm{ch}"'] & & K_j(\mathcal{A}) \arrow[d, "\mathrm{ch}"] \\
\mathrm{H}_j(G;\beta^1 G)\otimes_{\mathbb{Z}}\mathbb{C} \arrow[r] & \mathrm{HP}_j(\mathcal{H}(G)) \arrow[r, "\imath_*"] & \mathrm{HP}_j(\mathcal{C}(G))
\end{tikzcd}$$

with $j = 0, 1$. In this diagram, $K_j^{\mathrm{top}}(G)$ denotes the topological K-theory of G, $K_j(\mathcal{A})$ denotes K-theory for the C^*-algebra $\mathcal{A}$. In addition, $\mathrm{HP}_j(\mathcal{H}(G))$ denotes periodic cyclic homology of the algebra $\mathcal{H}(G)$, and $\mathrm{HP}_j(\mathcal{C}(G))$ denotes periodic cyclic homology of the topological algebra $\mathcal{C}(G)$. For periodic cyclic homology, see [12, 2.4].

The Baum-Connes assembly map μ is an isomorphism [3, 15]. The map

$$\mathrm{H}_*(G;\beta^1 G)\otimes_{\mathbb{Z}}\mathbb{C} \longrightarrow \mathrm{HP}_*(\mathcal{H}(G))$$

is an isomorphism [14, 20]. The map $\imath_*$ is an isomorphism by [3, 6]. The right hand Chern character is constructed in [7] and is an isomorphism after tensoring over $\mathbb{Z}$ with $\mathbb{C}$ [7, Theorem 3]. The left hand Chern character is the unique map for which the diagram is commutative.

B The Bernstein spectrum

Let G be the group of F-points of a connected reductive algebraic group defined over F. We consider pairs (L,σ) where L is a Levi subgroup of a parabolic subgroup of G, and σ is an irreducible supercuspidal representation of L. We say two such pairs (L_1,σ_1), (L_2,σ_2) are *inertially equivalent* if there exist $g \in G$ and an unramified quasicharacter χ of L_2 such that

$$L_2 = L_1^g \quad \text{and} \quad \sigma_1^g \cong \sigma_2 \otimes \chi.$$

Here, $L_1^g := g^{-1}L_1 g$ and $\sigma_1^g(x) = \sigma_1(gxg^{-1})$ for all $x \in L_1^g$. We write $[L,\sigma]_G$ for the inertial equivalence of the pair (L,σ) and $\mathfrak{B}(G)$ for the set of all inertial equivalence classes. The set $\mathfrak{B}(G)$ is the *Bernstein spectrum* of G. We will write $\mathfrak{s} \in \mathfrak{B}(G)$.

The Hecke algebra $\mathcal{H}(G)$ is a unital $\mathcal{H}(G)$-module via left multiplication, and admits the canonical Bernstein decomposition as a purely algebraic direct

sum of two-sided ideals:

$$\mathcal{H}(G) = \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} \mathcal{H}(G)^{\mathfrak{s}}.$$

This determines the canonical Bernstein decomposition of the reduced C^*-algebra as a C^*-direct-sum of two-sided C^*-ideals:

$$\mathcal{A} = \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} \mathcal{A}^{\mathfrak{s}}.$$

Now C^*-direct sums are respected by the K-theory of C^*-algebras, and we have

$$K_j(\mathcal{A}) = \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} K_j(\mathcal{A}^{\mathfrak{s}}) \tag{58}$$

with $j = 0, 1$. The abelian groups $K_j(\mathcal{A}^{\mathfrak{s}})$ are finitely generated free abelian groups, see [17].

We will define $\mathrm{H}_{\mathrm{ev/odd}}(G; \beta^1 G)^{\mathfrak{s}}$ as the pre-image of $K_j(\mathcal{A}^{\mathfrak{s}})$ via the commutative diagram in Appendix A:

$$\mathrm{H}_{\mathrm{ev}}(G; \beta^1 G)^{\mathfrak{s}} \cong K_0(\mathcal{A}^{\mathfrak{s}}), \quad \mathrm{H}_{\mathrm{odd}}(G; \beta^1 G)^{\mathfrak{s}} \cong K_1(\mathcal{A}^{\mathfrak{s}}). \tag{59}$$

C The formula for the rank

Let $\mathfrak{s}$ be a point in the Bernstein spectrum $\mathfrak{B}(G)$, so that $\mathfrak{s} = [L, \sigma]_G$. We have

$$L = \mathrm{GL}(m_1)^{e_1} \times \cdots \times \mathrm{GL}(m_r)^{e_r}$$

with $m_1 e_1 + \cdots + m_r e_r = N$. The numbers $e_1, \ldots, e_r$ are called the *exponents* of $\mathfrak{s}$, as in [6]. According to [6, Lemma 3.2], we then have

$$\operatorname{rank} K_j(\mathcal{A}^{\mathfrak{s}}) = 2^{r-1}\beta(e_1)\cdots\beta(e_r) \tag{60}$$

where

$$\beta(e) = \sum 2^{\kappa(\pi)-1}.$$

In this formula, π is a partition of e, the sum is over all partitions of e, and $\kappa(\pi)$ is the number of unequal parts of π. For example, if π is the partition $1+1+1+3+3+3+3+7+9$ of 31 then $\kappa(\pi) = 4$.

The ranks of the finitely generated abelian groups $\mathrm{H}_{\mathrm{ev}/odd}(G; \beta^1 G)^{\mathfrak{s}}$ are given by

$$\operatorname{rank} \mathrm{H}_{\mathrm{ev/odd}}(G; \beta^1 G)^{\mathfrak{s}} = 2^{r-1}\beta(e_1)\cdots\beta(e_r). \tag{61}$$

D Invariants attached to $\mathfrak{s}$

We write the supercuspidal representation σ of the Levi subgroup

$$M \cong \prod_{i=1}^{q} \prod_{j=1}^{c_i} \mathrm{GL}(N_{i,j}, F)$$

as a vector $\sigma = (\sigma_{1,1}, \ldots, \sigma_{1,c_1}, \sigma_{2,1}, \ldots, \sigma_{2,c_2}, \ldots, \sigma_{q,1}, \ldots, \sigma_{q,c_q})$ where $\sigma_{i,j}$ is an irreducible supercuspidal representation of $\mathrm{GL}(N_{i,j}, F)$, and for each $i \in \{1, \ldots, q\}$, the representations $\sigma_{i,j}$ $(1 \leq j \leq c_i)$ admit the same endo-class. At the same time, for all $1 \leq j \leq c_i$ and $1 \leq j' \leq c_{i'}$, the representations $\sigma_{i,j}$ and $\sigma_{i',j'}$ have distinct endo-classes if $i' \neq i$. This implies that, for a given i, in the construction of Bushnell-Kutzko, all the representations $\sigma_{i,j}$ $(1 \leq j \leq c_i)$ may be assumed to correspond to the same field extension E_i of F. Let $e(E_i|F)$ denote the ramification index of E_i over F. Then the parahoric subgroup $J^{\mathfrak{s}}$ only depends on the integers $N_{i,j}$, c_i and $e(E_i|F)$ (see Definition 6).

For supercuspidal representations, the parahoric subgroup is always the same one, say $\mathrm{GL}(N, \mathfrak{o}_F)$; when $q = 1$ (that is, only one endo-class), the parahoric is given by the integers $N_{1,1}$, ..., N_{1,c_1}, which are the sizes of the blocks of M. In the general case, the parahoric subgroup depends on the sizes of the blocks of M, of the block decomposition defined by the endo-classes (that is, those corresponding to the Levi subgroup $\bar{M} \cong \prod_{i=1}^{q} \mathrm{GL}(\bar{N}_i)$, with $\bar{N}_i = \sum_{j=1}^{c_i} N_{i,j}$) and on the ramification indices.

References

[1] Adler, J. and Roche, A.: An intertwining result for p-adic groups, *Canad. J. Math.* **52** (2000) 449–467.

[2] Baum, P., Connes, A. and Higson, N.: Classifying space for proper actions and K-theory of group C^*-algebras, *Contemporary Math.* **167** (1994) 241–291.

[3] Baum, P., Higson, N. and Plymen, R.J.: A proof of the Baum-Connes conjecture for p-adic $\mathrm{GL}(n)$, *C. R. Acad. Sci. Paris* **325** (1997) 171-176.

[4] Baum, P., Higson, N. and Plymen, R.J.: Representations of p-adic groups: a view from operator algebras, *Proc. Symp. Pure Math.* **68** (2001) 111–149.

[5] Bernstein, J. (rédigé par P. Deligne): Le "centre" de Bernstein. *Représentations des groupes réductifs sur un corps local.* Hermann, Paris (1984) 1–32.

[6] Brodzki, J. and Plymen, R.J.: Complex structure on the smooth dual of GL(n), *Documenta Math.* **7** (2002) 91–112.

[7] Brodzki, J. and Plymen, R.J.: Chern character for the Schwartz algebra of p-adic GL(n), *Bull. London Math. Soc.* **34** (2002) 219–228.

[8] Bushnell, C.J. and Kutzko, P.C.: The admissible dual of GL(N) via compact open subgroups, *Annals of Math. Study* **129** (1993) Princeton University Press.

[9] Bushnell, C.J. and Kutzko, P.C.: Smooth representations of reductive p-adic groups: structure theory via types, *Proc. London Math. Soc.* **77** (1998) 582–634.

[10] Bushnell, C.J. and Kutzko, P.C.: Semisimple types in GL_n, *Compos. Math.* **119** (1999) 53–97.

[11] Bushnell, C.J. and Kutzko, P.C.: Types in reductive p-adic groups: the Hecke algebra of a cover, *Proc. Amer. Math. Soc.* **129** (2001) 601–607.

[12] Cuntz, J., Skandalis, G. and Tsygan, B.: *Cyclic homology in noncommutative geometry*, EMS 121, Springer-Verlag, Berlin 2004.

[13] Gelfand, S.I. and Manin, Yu. I.: *Homological algebra*, Springer-Verlag, Berlin 1999.

[14] Higson, N. and Nistor, V.: Cyclic homology of totally disconnected groups acting on buildings, *J. Functional Analysis* **141** (1996) 466–485.

[15] Lafforgue, V.: K-théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes, *Invent. Math.* **149** (2002) 1–95.

[16] Paskunas, V.: Unicity of types for supercuspidal representations of GL_N, *Proc. London Math. Soc.* **91** (2005) 623–654.

[17] Plymen, R.J.: Reduced C^*-algebra of the p-adic group GL(n), *J. Functional Analysis* **72** (1987) 1–12.

[18] Roche, A.: Types and Hecke algebras for principal series representations of split reductive p-adic groups, *Ann. scient. Éc. Norm. Sup.* **31** (1998) 361–413.

[19] Ronan, M.: *Lectures on buildings*, Academic press (1989).

[20] Schneider, P.: Equivariant homology for totally disconnected groups, *J.Algebra* **203** (1998) 50–68.

[21] Tits, J.: Reductive groups over local fields, in *Automorphic forms, representations and L-functions*, *Proc. Symp. Pure Math.* **33** (1979), part 1, 29–69.

Anne-Marie Aubert, Institut de Mathématiques de Jussieu, U.M.R. 7586 du C.N.R.S., 175 rue du Chevaleret 75013 Paris, France.
Email: aubert@math.jussieu.fr

Samir Hasan, Department of Pure Mathematics, Faculty of Sciences, University of Damascus, Damascus, S.A.R., SYRIA.
Email: samir.hasan@gmail.com

Roger Plymen, School of Mathematics, Manchester University, M13 9PL, England.
Email: plymen@manchester.ac.uk